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Workshop on mathematical analysis

Part 1

Study guide

Recommended by the Methodical Council
of Igor Sikorsky Kyiv Polytechnic Institute
as a study guide for students of the specialty
121 "Software engineering", specialization
"Software for computer and information search systems"

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The English-language educational material of the practicum corresponds to the Syllabus for the academic discipline "Mathematical Analysis" for the first semester. It covers the following sections: "Introduction to Mathematical Analysis", "Differential Calculus of Functions of One Variable" and "Integral Calculus of Functions of One Variable". The workshop is divided into 18 educational topics, each of which has a separate practical session. At the beginning of each practical lesson, a concise theoretical material is given, which contains definitions of basic concepts, formulation of theorems and formulas necessary for solving problems. The theoretical material of each topic is illustrated in detail with solved examples and methodical recommendations for their implementation. In each topic, examples and problems are presented for conducting classroom classes and independent work of students. At the end of each topic, there are answers to the tasks or instructions for completing them. At the end of the workshop there is a list of used and recommended literature.

For foreign students of higher education in the specialty 121 "Software Engineering", who are trying to master the course of mathematical analysis with high quality, want to form the necessary skills and abilities from the application of the acquired theoretical knowledge, as well as for the teachers working with them in order to improve their pedagogical skills.

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PREFACE

The rapid development of modern computer systems and technologies has significantly expanded the boundaries of the successful application of mathematical methods during the study of theoretical and applied problems. The correct formulation of any engineering and technical problem, the construction of its mathematical model, followed by the development of an algorithm and a software product requires an IT specialist to have in-depth knowledge of fundamental disciplines. The course of mathematical analysis is the basis of the general mathematical training of an IT specialist or engineer at a technical university.

According to the requirements of the standard of higher education in the specialty 121 “Software engineering”, the study of the discipline “Mathematical analysis” contributes to the formation of the following general and professional competences in students of higher education:

– ability to abstract thinking, analysis and synthesis;

– ability to apply knowledge in practical situations;

– the ability to learn and master modern knowledge;

– ability to algorithmic and logical thinking;

– the ability to apply the acquired fundamental mathematical knowledge for the development of calculation methods in the creation of multimedia and information-search systems.

The acquired knowledge provides the following program learning outcomes:

– know and apply relevant mathematical concepts, methods of domain, system and object-oriented analysis and mathematical modeling for software development;

– select initial data for design, guided by formal requirements description and modeling methods;

– know and be able to use fundamental mathematical tools when constructing algorithms and developing modern software.

An important place for the successful mastering of sections of mathematical analysis is given to practical classes. Active and systematic work on them under the guidance of a teacher contributes to the qualitative assimilation of theoretical material, gives students practical skills for independent work and scientific research.

There are now a number of good collections of mathematical analysis problems. However, their content often does not agree with the curricula of some IT specialties, and due to the large amount of material of such problem books, it is difficult for first-year students to find their way around them. In addition, most problem books do not contribute to students’ independent work, as they do not contain solutions to typical problems.

The offered practicum in mathematical analysis is intended for the organization of high-quality classroom and independent work of students during the study of the discipline "Mathematical Analysis". The workshop presents sections of mathematical analysis, which are taught in the first semester to students of specialty 121 "Software engineering" of the Faculty of Applied Mathematics of the National University of Ukraine "Ihor Sikorskyi Kyiv Polytechnic Institute".

The content of the workshop corresponds to the program of the educational discipline (Syllabus) and covers the following sections: "Introduction to mathematical analysis", "Differential calculus of functions of one variable" and "Integral calculus of functions of one variable".

18 topics of practical classes are distinguished in the workshop. At the beginning of each topic, theoretical material is briefly presented, which contains definitions of basic concepts, formulation of theorems and formulas necessary for solving problems. The theoretical material of the topic is illustrated with detailed examples and methodical recommendations for their implementation. Each topic has a sufficient number of examples and problems for conducting classroom classes and independent work of students. At the end of the topics, the answers to the tasks or instructions for their implementation are given.

When working with the manual, the student must first of all study the theoretical material related to the relevant topic, analyze in detail the given solutions to the examples, and then complete the tasks proposed for the topic. The theoretical material is presented in its entirety in the basic textbook [1].

The educational material presented in the workshop was successfully used in practical classes by students of higher education during the distance learning process at the Department of Computer Systems Software, Faculty of Applied Mathematics of Igor Sikorsky Kyiv Polytechnic Institute.

The workshop is prepared for students of higher education in the specialty 121 "Software engineering" who seek to master the course of mathematical analysis with high quality, who wish to form the necessary skills and abilities from the application of acquired theoretical knowledge, as well as for teachers who work with them in order to improve their pedagogical skills.

This practicum will be useful for students of other technical majors studying higher mathematics.

The authors

Section I. INTRODUCTION TO MATHEMATICAL ANALYSIS
Topic I. Induction, sets and functions

PRACTICAL LESSON 1
Method of mathematical induction

In mathematics, there are often statements that depend on a natural number. To prove such statements, the method of mathematical induction is used, which is based on such a theorem.

Theorem (*Principle of mathematical induction*). *Let it be necessary to prove some* ***statement (hypothesis)*** $P(n)$*, where* $n \in \mathbb{N}$*. If the conditions are met:*

1) *the statement is valid according to* $n = 1$ ***(the basis of induction)***;

2) *if for any* $k \in \mathbb{N}$ *is true* $P(k)$ *– the statement for* $n = k + 1$ ***(inductive transition)*** *is true, then the statement* $P(n)$ *is valid for any value* $n \in \mathbb{N}$.

Remark. Sometimes it is necessary to prove the truth of the statement $P(n)$ for whole values n, starting from some number $n_0 \in \mathbb{Z}$. In this case, the basis of induction will be the verification of the truth of the statement $P(n)$ by $n = n_0$.

Let's consider examples that illustrate the method of mathematical induction. In some of them, we will use Newton's binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k} .$$

Example 1.1. Prove that the expression $7^{n+2} + 8^{2n+1}$ is divisible by 57 for any $n \geq 0$.

The solution. Currently, we have the statement $P(n)$: "$7^{n+2} + 8^{2n+1}$ is divisible by 57" for $n \geq 0$.

Induction base. We make sure that $n = 0$ the following statement $P(0)$ is true because $7^{0+2} + 8^{0+1} = 49 + 8 = 57$ it is divisible by 57.

Inductive transition. We assume that $P(k)$ it is true for an arbitrary non-negative integer *k*, i.e. the expression $7^{k+2} + 8^{2k+1}$ is divisible by 57. Let us now prove that under the condition of the hypothesis $P(k)$ being fulfilled, the statement $P(k+1)$ will also be true, i.e. $7^{(k+1)+2} + 8^{2(k+1)+1}$ it is divisible by 57. We have:

$$7^{(k+1)+2} + 8^{2(k+1)+1} = 7^{k+3} + 8^{2k+3} = 7 \cdot 7^{k+2} + 8^2 \cdot 8^{2k+1} =$$

$$= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} = 7(7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1}.$$

Therefore, both conditions of the theorem are fulfilled, the statement $P(n)$ is valid for any value of $n \geq 0$.

Example 1.2. Set the expression for the sum S_n if

$$S_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)}.$$

The solution. Let's consider and test the *hypothesis* that for an arbitrary natural number n the sought sum is $S_n = \frac{n}{n+1}$.

Induction base. For $n = 1$ we have the sum $S_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$ – the correct equality.

Inductive transition. Suppose that the written formula is true for $n = k$, i.e. $S_k = \frac{k}{k+1}$. Based on this formula, it is necessary to prove that $S_{k+1} = \frac{k+1}{k+2}$. Let's check the second condition of the theorem:

$$S_{k+1} = S_k + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} =$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}.$$

Thus, it is proved that the sought sum S_n is calculated by the formula $S_n = \frac{n}{n+1}$ for an arbitrary natural number n.

Consider how the formula for the original *hypothesis* in this problem is derived.

Example 1.3. Using the method of mathematical induction, prove the formula:

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2. \tag{1.1}$$

The solution. In this problem, it is necessary to check the hypothesis $P(n)$, which is that equality (1.1) holds.

Induction base. For $n = 1$ formula (1.1) is valid: $1^3 = 1^2$.

Inductive transition. Now suppose that the written formula is true for $n = k$, i.e.^

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = (1 + 2 + 3 + \cdots + k)^2.$$

It is necessary to prove the statement of the second condition of the theorem, i.e. to show that the formula is valid according to $n = k + 1$:

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = [(1 + 2 + 3 + \cdots + k) + (k+1)]^2 . \qquad (1.2)$$

Using the assumption, we write the left part of the expression (1.2) as follows:

$$(1 + 2 + 3 + \cdots + k)^2 + (k+1)^3 .$$

The right-hand side of expression (1.2) is transformed by the formula of the square of the sum of two terms $a = 1 + 2 + 3 + \cdots + k$ and $b = k + 1$:

$$[(1 + 2 + 3 + \cdots + k) + (k+1)]^2 = (1 + 2 + 3 + \cdots + k)^2 + 2(1 + 2 + 3 + \cdots + k)(k+1) +$$

$$+(k+1)^2 = (1 + 2 + 3 + \cdots + k)^2 + 2\frac{(1+k)}{2}k(k+1) + (k+1)^2 =$$

$$= (1 + 2 + 3 + \cdots + k)^2 + k(k+1)^2 + (k+1)^2 = (1 + 2 + 3 + \cdots + k)^2 + (k+1)^3.$$

We see that the left and right parts of expression (1.2) are the same. So, by the method of mathematical induction, we proved that formula (1.1) is valid for any $n \in \mathbb{N}$.

What formula was used in the transformation process?

Example 1.4. Prove the inequality

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n} \qquad (1.3)$$

by the method of mathematical induction for $n > 1$.

The solution. Let's check the first condition of the theorem (*base of induction*) according to $n = 2$:

$$1 + \frac{1}{\sqrt{2}} > \sqrt{2} .$$

This inequality holds because $\sqrt{2} + 1 > 2$.

Inductive transition. Assume that the given inequality (1.3) is valid according to $n = k$:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k} .$$

Now we need to prove that relation (1.3) is also valid for $n = k + 1$, i.e.:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} . \qquad (1.4)$$

To prove the formula (1.4), using the assumption, we replace the first k terms in its left part with a smaller value $\sqrt{k}$; we get the inequality

$$\sqrt{k}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}.$$

It is obvious that such a replacement reduced the left side of inequality (1.4). If after that the obtained inequality is valid, then the inequality (1.4) will also be fulfilled (because its left part is greater than $\sqrt{k}$). Let's check whether the last inequality holds:

$$\sqrt{k}+\frac{1}{\sqrt{k+1}}>\sqrt{k+1}\Rightarrow\sqrt{k(k+1)}+1>k+1\Rightarrow\sqrt{k(k+1)}>k\Rightarrow\sqrt{k+1}>\sqrt{k}.$$

After identical transformations, an obvious inequality is obtained, which is valid for arbitrary natural numbers ($k>1$), therefore, inequality (1.4) is also satisfied, and together with it, the given inequality (1.3).

Example 1.5. Prove the inequality by the method of mathematical induction

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2n-1}{2n}<\frac{1}{\sqrt{2n+1}} \tag{1.5}$$

for $n\geq 1$.

The solution. Let's check the first condition of the theorem (*base of induction*) by $n=1$. It is obvious that the obtained inequality $\frac{1}{2}<\frac{1}{\sqrt{3}}$ holds.

Let us now check the second condition of the theorem (*inductive transition*). Assume that inequality (1.5) is satisfied by $n=k$:

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2k-1}{2k}<\frac{1}{\sqrt{2k+1}};$$

and we will prove its validity by $n=k+1$, i.e.:

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\ldots\cdot\frac{2k-1}{2k}\cdot\frac{2k+1}{2(k+1)}<\frac{1}{\sqrt{2k+3}}. \tag{1.6}$$

Let's replace the first k factors in the left part of the expression (1.6) with the right part of the assumption; as a result, we get the following inequality:

$$\frac{1}{\sqrt{2k+1}}\cdot\frac{2k+1}{2(k+1)}<\frac{1}{\sqrt{2k+3}}.$$

Thus, if this inequality is valid, then inequality (1.6) will also hold. Let's check the validity of the obtained inequality:

$$\frac{\sqrt{2k+1}}{2(k+1)} < \frac{1}{\sqrt{2k+3}} \Rightarrow 2(k+1) > \sqrt{(2k+3)(2k+1)} \Rightarrow$$

$$\Rightarrow 4k^2 + 8k + 4 > 4k^2 + 8k + 3 \Rightarrow 4 > 3.$$

After identical transformations, an obvious inequality is obtained, which is valid for arbitrary natural numbers ($n \geq 1$). Therefore, the inequality (1.6) is also fulfilled, and together with it, the given inequality (1.5).

Example 1.6. Using the method of mathematical induction, prove the trigonometric identity:

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos nx = \frac{\sin\left(\frac{2n+1}{2} \cdot x\right)}{2\sin\left(\frac{x}{2}\right)}. \quad (1.7)$$

The solution. In this problem, the hypothesis has already been proposed, and it remains only to be tested.

Induction base. Let's check the first condition of the theorem by $n = 1$. That is, it is necessary to show that the following equality is fair:

$$\frac{1}{2} + \cos x = \frac{\sin\left(\frac{3x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}.$$

Let's transform the right-hand side of this equality:

$$\frac{\sin\left(\frac{3x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \frac{\sin\left(\frac{x}{2} + x\right)}{2\sin\left(\frac{x}{2}\right)} = \frac{\sin(x)\cdot\cos\left(\frac{x}{2}\right) + \cos(x)\cdot\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} = \frac{2\sin\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} +$$

$$+\frac{1}{2}\cos(x) == \cos^2\left(\frac{x}{2}\right) + \frac{1}{2}\cos(x) = \cos^2\left(\frac{x}{2}\right) + \frac{1}{2}\left(1 - 2\sin^2\left(\frac{x}{2}\right)\right) = \cos x + \frac{1}{2}.$$

As we can see, equality (1.7) is true for $n = 1$.

Now let's check the second condition of the theorem (*inductive transition*).

Assume that the given identity is fulfilled by $n = k$:

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos kx = \frac{\sin\left(\frac{2k+1}{2}\cdot x\right)}{2\sin\left(\frac{x}{2}\right)}.$$

We will prove its validity by $n = k+1$:

$$\frac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots + \cos kx + \cos(k+1)x = \frac{\sin\left(\frac{2k+3}{2}\cdot x\right)}{2\sin\left(\frac{x}{2}\right)}. \quad (1.8)$$

Using the assumption on the left side of equality (1.8), we obtain the following trigonometric identity:

$$\frac{\sin\left(\frac{2k+1}{2}\cdot x\right)}{2\sin\left(\frac{x}{2}\right)} + \cos(k+1)x = \frac{\sin\left(\frac{2k+3}{2}\cdot x\right)}{2\sin\left(\frac{x}{2}\right)}.$$

After identical transformations, this expression will take the following form:

$$2\sin\left(\frac{x}{2}\right)\cos[(k+1)x] = \sin\left(\frac{2k+3}{2}\cdot x\right) - \sin\left(\frac{2k+1}{2}\cdot x\right).$$

In the left part, we will use the well-known formula

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta),$$

we will get:

$$2\sin\left(\frac{x}{2}\right)\cos[(k+1)x] = \sin\left[\frac{x}{2} + (k+1)x\right] + \sin\left[\frac{x}{2} - (k+1)x\right] =$$

$$= \sin\left[\frac{(3+2k)x}{2}\right] - \sin\left[\frac{(2k+1)x}{2}\right].$$

It can be seen that equality (1.8) is correct. Thus, the trigonometric identity (1.7) is proved for all natural n.

Example 1.7. Prove the inequality by the method of mathematical induction

$$2^{n-1}(a^n + b^n) > (a+b)^n \quad (1.9)$$

for $n \geq 2,\ a+b > 0,\ a \neq b$.

The solution. Let’s check the first statement of the theorem by $n = 2$. That is, it is necessary to show that the following inequality is fair: $2(a^2 + b^2) > (a + b)^2$. After identical transformations, we obtain the inequality $(a - b)^2 > 0$, which always holds, except in the case of $a = b$.

Now let’s check the second statement. Suppose that the given inequality is fulfilled by $n = k$, i.e.:

$$2^{k-1}(a^k + b^k) > (a + b)^k . \tag{1.10}$$

Therefore, using inequality (1.10), it is necessary to prove the validity of inequality (1.9) for $n = k + 1$. Let’s write down the inequality that needs to be proved:

$$2^k(a^{k+1} + b^{k+1}) > (a + b)^{k+1} . \tag{1.11}$$

Multiply both parts of expression (1.10) by $a + b > 0$, as a result we get:

$$2^{k-1}(a^k + b^k)(a + b) > (a + b)^{k+1} . \tag{1.12}$$

Let’s prove that

$$2^k(a^{k+1} + b^{k+1}) > 2^{k-1}(a^k + b^k)(a + b) . \tag{1.13}$$

Let’s reduce by a factor 2^{k-1} and open the parentheses:

$$2a^{k+1} + 2b^{k+1} > a^{k+1} + b^{k+1} + ba^k + ab^k \Rightarrow a^{k+1} + b^{k+1} > ba^k + ab^k \Rightarrow$$

$$\Rightarrow (a^k - b^k)(a - b) > 0 . \tag{1.14}$$

We will show that the inequality (1.14) is correct under the restrictions stated in the condition of the problem.

A. If $a > b$, then $a^k > b^k$, therefore, on the left in inequality (1.14) we have the product of two positive numbers. Therefore, inequality (1.14) is true in this case.

B. If $a < b$, then $a^k < b^k$, therefore, on the left in inequality (1.14) we have the product of two negative numbers. Therefore, inequality (1.14) also holds true in this case.

Since the inequality (1.14) is correct, and together with it the inequality (1.11), we thereby completed the proof of the given inequality (1.9).

Example 1.8. Prove the following inequality using the method of mathematical induction:

$$n! < e\left(\frac{n}{2}\right)^n , \; n \geq 1 . \tag{1.15}$$

The solution. Let's check the *basis of induction* by $n=1$. We have: $1<\frac{e}{2}$ – this inequality holds. Now let's check the second statement of the theorem. Suppose that the given inequality is fulfilled by $n=k$, i.e.:

$$k!<e\left(\frac{k}{2}\right)^k. \tag{1.16}$$

Now, using inequality (1.16), it is necessary to prove the validity of inequality (1.15) for $n=k+1$. Let's write down the inequality that needs to be proved:

$$(k+1)!<e\left(\frac{k+1}{2}\right)^{k+1}. \tag{1.17}$$

We multiply both parts of the expression (1.16) by $(k+1)$, as a result we get:

$$(k+1)!<e\left(\frac{k}{2}\right)^k(k+1). \tag{1.18}$$

Let's replace the left part in inequality (1.17) with the right part from (1.18), which is greater than $(k+1)!$. It is obvious that if the new inequality is correct, then inequality (1.17) will also be correct. Let's write down this new inequality and check if it holds:

$$e\left(\frac{k}{2}\right)^k(k+1)<e\left(\frac{k+1}{2}\right)^{k+1}\Rightarrow 2<\left(\frac{k+1}{k}\right)^k\Rightarrow 2<\left(1+\frac{1}{k}\right)^k. \tag{1.19}$$

To prove inequality (1.19), we will open its right-hand side using Newton's binomial formula and keep the first three terms of this expansion:

$$\left(1+\frac{1}{k}\right)^k=1^k+C_k^1\cdot 1^{k-1}\cdot\left(\frac{1}{k}\right)^1+C_k^2\cdot 1^{k-2}\cdot\left(\frac{1}{k}\right)^2+\cdots=2+\frac{k!}{2!(k-2)!k^2}+\cdots=2+\frac{k-1}{2k}+\cdots.$$

Since all terms of this expansion are positive, it is obvious that the right-hand side of the inequality (1.19) is greater than 2. Thus, we have shown that the inequality (1.17) is true, and therefore, by the method of mathematical induction, the original inequality (1.15) is correct.

Tasks for classroom and independent work

I. Using the method of mathematical induction, prove the identities.

1.1. $\frac{1^2}{1\cdot 3}+\frac{2^2}{3\cdot 5}+\cdots+\frac{n^2}{(2n-1)\cdot(2n+1)}=\frac{n(n+1)}{2\cdot(2n+1)}$.

1.2. $1^2+2^2+3^2+\cdots+n^2=\frac{n(n+1)(2n+1)}{6}$.

1.3. $1\cdot 2\cdot 3+2\cdot 3\cdot 4+3\cdot 4\cdot 5+\cdots+n(n+1)(n+2)=\frac{n(n+1)(n+2)(n+3)}{4}$.

1.4. $\sin x+\sin 2x+\sin 3x+\cdots+\sin nx=\frac{\sin\left(\frac{(n+1)x}{2}\right)\cdot\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}$.

1.5. $\sin x+2\sin 2x+3\sin 3x+\cdots+n\sin nx=\frac{(n+1)\sin nx-n\sin[(n+1)x]}{4\sin^2\left(\frac{x}{2}\right)}$.

1.6. $\cos x+2\cos 2x+3\cos 3x+\cdots+n\cos nx=\frac{(n+1)\cos nx-n\cos[(n+1)x]-1}{4\sin^2\left(\frac{x}{2}\right)}$.

1.7. $\frac{1}{2}\operatorname{tg}\left(\frac{x}{2}\right)+\frac{1}{2^2}\operatorname{tg}\left(\frac{x}{2^2}\right)+\cdots+\frac{1}{2^n}\operatorname{tg}\left(\frac{x}{2^n}\right)=\frac{1}{2^n}\operatorname{ctg}\left(\frac{x}{2^n}\right)-\operatorname{ctg}(x)$; $x\neq\pi m$.

II. Using the method of mathematical induction, prove inequalities.

1.8. $2^n>n^2,\ \ n\geq 1$.

1.9. $5^n+7^n>2\cdot 6^n,\ \ n\geq 2$.

1.10. $n^{n+1}>(n+1)^n,\ \ n\geq 3$.

1.11. $n!<\left(\frac{n+1}{2}\right)^n,\ \ n>1$.

1.12. $n!>\left(\frac{n}{e}\right)^n,\ \ n\geq 1$.

1.13. $(1+\alpha)^n>1+n\alpha,\ \ \alpha>-1, \alpha\neq 0, n\geq 2$.

1.14. $\left(1+\frac{1}{n}\right)^n<e,\ \ n\geq 1$.

1.15. $\left(1+\frac{1}{n}\right)^{n+1}>e,\ \ n\geq 1$.

III. Using the method of mathematical induction, prove that the expression.

1.16. $n^5 - 5n^3 + 4n$ is divisible by 120 for any $n \geq 0$.

1.17. $4^{n+1} + 5^{2n-1}$ is divisible by 21 for any $n \in \mathbb{N}$.

1.18. $11^{n+1} + 12^{2n-1}$ is divisible by 133 for any $n \in \mathbb{N}$.

1.19. Using the method of mathematical induction, prove Newton's binomial formula.

PRACTICAL LESSON 2
Sets. Actions on sets. Mapping of sets. Images and preimage. Three types of reflections

2.1. Sets. Actions on sets

A ***set*** is a specific collection of objects of any nature that have some common feature. Objects that form a set are called its ***elements***.

Sets are denoted by uppercase Latin letters: *A*, *B*, *C*, ..., and elements of sets – lowercase: *a*, *b*, *c*, ... (without indices or with indices). The fact that the object *a* is an element of the set *M* is written as follows: $a \in M$ (read: "*a* belongs to the set *M*", "*a* is an element of the set *M*", "the set *M* contains an element *a*", "*a* is part of the set *M*"). The symbol "$\in$" is a ***sign of belonging*** to a set element. In order to emphasize that some element *a* does not belong to the set *M*, the notation $a \notin M$ or $a \,\overline{\in}\, M$ is used.

A set can be defined by a ***list*** of its elements, for example, $M = \{a, b, c, ..., x, y\}$ or $A = \{a_1, a_2, ..., a_n\}$. In the general case, a set is given by means of a ***characteristic property*** (condition) $P(x)$, which is satisfied by all the elements of this set and only by them. In this case, record: $M = \{x : P(x)\}$. This expression reads as follows: "the set *M* is the set of all such elements *x* for which the property *P* is fulfilled", where $P(x)$ denotes the property possessed by the elements of the set *M* and only by them. Sometimes a vertical dash is used instead of a colon.

For some sets in mathematics, generally accepted notations are used: *N* – set of natural numbers; *Z* is a set of integers; *Q* is a set of rational numbers; *R* is a set of real numbers; *C* is a set of complex numbers; $[a; b]$ – numerical interval (segment); $(a; b)$ – numeric interval.

Definition. *The set A is called a* ***subset*** *of the set B (denoted by* $A \subseteq B$ *or* $B \supseteq A$*) if every element of the set A belongs to the set B:*

$$A \subseteq B \Leftrightarrow \forall x \ (x \in A \Rightarrow x \in B).$$

The signs $\subseteq$ and $\supseteq$ are called ***inclusion*** *or* ***non-strict inclusion*** signs.

Definition. *Sets A and B are called* ***equal*** *if they consist of the same elements, that is, each element of the set A is an element of the set B and vice versa; denote* $A = B$ *:*

$$A = B \Leftrightarrow (x \in A \Rightarrow x \in B \ i \ x \in B \Rightarrow x \in A).$$

Thus, to prove the equality of sets, it is necessary to establish two inclusion of $A \subseteq B$ and $B \subseteq A$.

Example 2.1. Show that the sets $A = \{x : \cos x = -1\}$ and $B = \{x : x = \pi + 2\pi n, n \in \mathbb{Z}\}$ are coincide.

The solution. If $x \in A$, then x is a solution of the equation $\cos x = -1$. This means that x can be represented as $x = \pi + 2\pi n,\ n \in \mathbb{Z}$, and therefore $x \in B$. Thus, we have inclusion $A \subseteq B$.

If $x \in B$, that is $x = \pi + 2\pi n$, which is equivalent to $\cos x = -1$, that is $B \subseteq A$. Therefore, according to the definition, we have the equality $A = B$.

If $A \subseteq B$, but $A \neq B$, then we write $A \subset B$ and the set *A* is called a ***proper*** (or ***strict***) ***subset*** of the set *B*. The symbol $\subset$ (or $\supset$), unlike the sign $\subseteq$ (or $\supseteq$), is called the sign of ***strict inclusion***.

A set that includes all the considered sets is called a ***universal set*** and is denoted by *U*. For example, in mathematical analysis, a universal set can be considered a set of all real numbers or the set of all points of *n*-dimensional space.

Consider operations on sets. These operations make it possible to construct new sets from given sets.

Definition. *The* ***union*** *or* ***sum*** *of the sets A and B (denoted by $A \cup B$) is called the set of those elements that belong to at least one of the sets A or B:*

$$A \cup B = \{x \mid x \in A \ or \ x \in B\}.$$

Definition. *The* ***intersection*** *or* ***product*** *of sets A and B (denoted by $A \cap B$) is a set consisting of those and only those elements that belong to sets A and B at the same time:*

$$A \cap B = \{x \mid x \in A \ and \ x \in B\}.$$

Definition. *The* ***difference*** *of the sets A and B (denoted by $A \setminus B$) is the set of all those and only those elements of the set A that do not belong to the set B:*

$$A \setminus B = \{x \mid x \in A \ and \ \notin B\}.$$

Definition. *The set*

$$A \oplus B \equiv (A \setminus B) \cup (B \setminus A)$$

is called the ***symmetric difference*** *of sets A and B.*

Definition. *The* ***complement*** *of the set A (denoted by $\overline{A}$) is called the set of all elements of the universal set that do not belong to the set A:*

$$\overline{A} = U \setminus A = \{x \mid x \in U \ and \ x \notin A\}.$$

Example 2.2. Given sets $U = \{x : |x| \le 4,\ x \in \mathbb{Z}\}$, $A = \{-2,-1,2,4\}$, $B = \{x : x^4 - x^3 - 7x^2 + 13x - 6 = 0\}$. Find sets: $A \cup B$, $A \cap B$, $A \setminus B$, $B \setminus A$, $\overline{B}$, $(A \oplus B) \cap \overline{B}$.

The solution. Let's write the sets *U* and *B* as a list of elements. Let's find the set of roots of the equation $x^4 - x^3 - 7x^2 + 13x - 6 = 0$. By selection, we make sure that $x = 1$ is the root of the equation. As a result of dividing this polynomial by $x - 1$ we get the polynomial $x^3 - 7x + 6$. Let's break it down into multiples:

$$x^3 - 7x + 6 = x(x^2 - 1) - 6(x - 1) = (x - 1)(x^2 + x - 6) = (x - 1)(x - 2)(x + 3).$$

So, we have the roots $x_{1,2} = 1,\ x_3 = 2,\ x_4 = -3$ of the given polynomial, and therefore the set $B = \{-3,1,2\}$. We have a universal set $U = \{-4,-3,-2,-1,0,1,2,3,4\}$.

We perform operations on sets:

$A \cup B = \{-2,-1,2,4\} \cup \{-3,1,2\} = \{-3,-2,-1,1,2,4\}$;

$A \cap B = \{-2,-1,2,4\} \cap \{-3,1,2\} = \{2\}$;

$A \setminus B = \{-2,-1,2,4\} \setminus \{-3,1,2\} = \{-2,-1,4\}$;

$B \setminus A = \{-3,1,2\} \setminus \{-2,-1,2,4\} = \{-3,1\}$;

$\overline{B} = \{-4,-3,-2,-1,0,1,2,3,4\} \setminus \{-3,1,2\} = \{-4,-2,-1,0,3,4\}$;

$(A \oplus B) \cap \overline{B} = \{-3,-2,-1,1,4\} \cap \{-4,-2,-1,0,3,4\} = \{-2,-1,4\}$.

2.2. Representation of sets. Images and preimage

Often, when solving problems, you have to deal with elements sets connected by some relationship. Let two sets X and Y.

Definition. *If each element* $x \in X$ *according to a known rule (law)* f *is matched with a certain element* $y \in Y$, *which is denoted by the symbol* $y = f(x)$, *then this correspondence (rule or operator) is called a* ***function*** *(****mapping****) acting from set* X *to set* Y *(Fig. 2.1).*

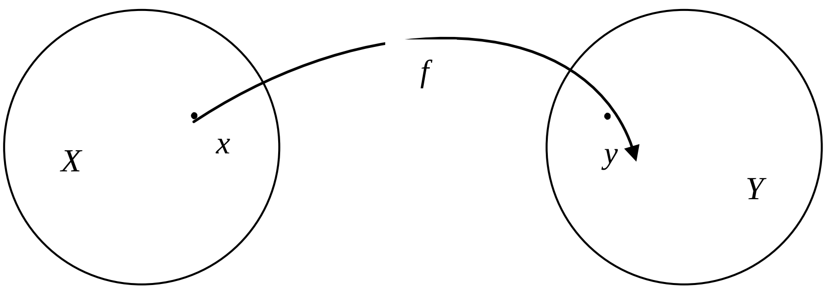

Fig. 2.1. The Function f Maps X to Y

This correspondence is indicated by the following symbols: $f : X \to Y$. At the same time, the set X is called the ***domain of definition*** (D_f) of the function $y = f(x)$.

Definition. *If f is a function from X to Y, we say that X is the **domain** of f and Y is the **codomain** of f. If $f(x) = y$, we say that y is the **image** of x and x is a **preimage** of y.*

Definition. *Let $f : X \to Y$ be a mapping from X to Y. The set E_f of all elements of the form $f(x) \in Y$ is called a **image of the mapping** f, i.e.:*

$$E_f = \{ f(x) \mid x \in X \} = f(X).$$

It is obvious that in the general case $f(X) \subset Y$.

Definition. The **image of an element** (or the value of the mapping at the point $x \in X$) when mapping is $f : X \to Y$ called an element $y \in Y$ such that $y = f(x)$.

Definition. *The **image of a subset** $A \subset X$ under mapping $f : X \to Y$ is called a subset $B \subset Y$ such that $B = \{ f(x) \mid x \in A \} = f(A)$.*

2.3. Three types of reflections: surjection, injection, and bijection

There are three types of reflections: ***surjection***, ***injection*** and ***bijection***.

Definition of surjection. *A mapping $f : X \to Y$ is called **surjective** (or **surrection**, or **set-to-set mapping**) if each element y of the set Y is the image of at least one element x of the set X, i.e. $\forall y \in Y\ \exists x \in X : y = f(x)$ (Fig. 2.2).*

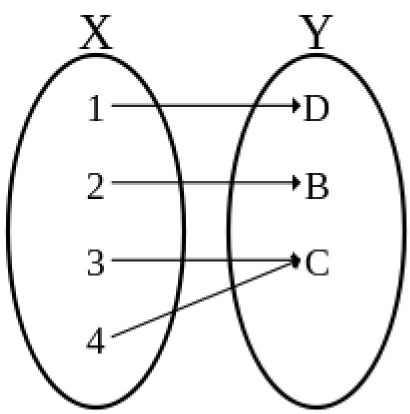

Fig. 2.2. A surjective function

Definition of injection. *The mapping $f : X \to Y$ of a set X into a set Y is called **injective** (or an **injunction**, or **one-to-one**) if different elements x of the set X go into different elements y of the set Y, i.e. $\forall x \in X\ \exists!\, y \in Y : y = f(x)$. At the same time, there may be elements y in the set Y that do not have corresponding elements $x \in X$ that "generate" them (Fig. 2.3).*

In other words, an ***injection*** is a relation between elements of two sets X and Y, which matches one element x from the first set X with one and only one element y from the second set Y.

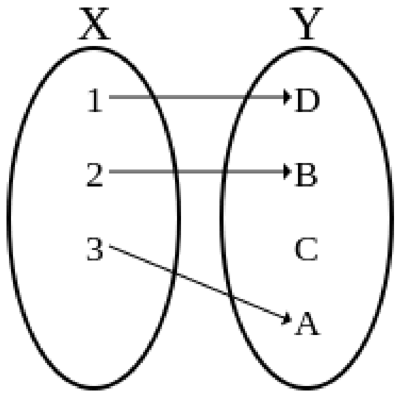

Fig. 2.3. A One-to-One Function

Definition of bijection. *The mapping $f: X \to Y$ is called **bijective** if it is both **injective** and **surjective**.*

In other words, this is a "one-to-one" mapping, and not a single element of the set X and the set Y was left "without an arrow" (Fig. 2.4).

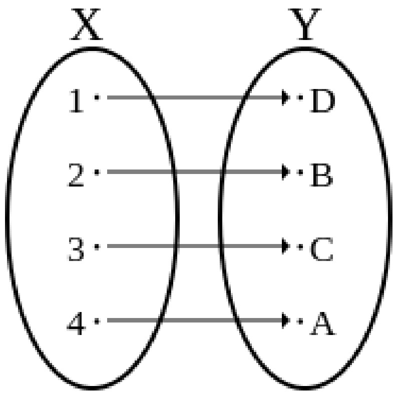

Fig. 2.4. The bijective function

Definition. *The set of all elements of the form $f^{-1}(y) \in X$ is called the **preimage of an element** $y \in Y$, where $f^{-1}(y) = \{x \in X \mid f(x) = y\}$.*

Definition. *The **preimage of a subset** $B \subset Y$ is called a set of the form $f^{-1}(B) = \{x \in X \mid f(x) \in B\}$.*

Definition. *If there is a **bijection** of the set A on the set B, then the sets A and B are called **equi-potent**. Equipotent sets are often called **equivalent**.*

Definition. *The sets that are equivalent to the set N of natural numbers are called **countable**, and the sets that are equivalent to the set R real numbers are called **continuous**.*

Let's consider these concepts using examples.

Example 2.3. Given mapping $f: R \to R_+$, where $f(x) = x^2$ and segment $x \in [-4,1]$. It is necessary to find:

a) the image of this segment $Z = f([-4,1])$ and

b) the preimage of the found image $f^{-1}(Z)$.

The solution. Note that the range of values of this mapping is the set of non-negative real numbers. In addition, the given mapping is surjective:

a) we find the image of this mapping: $f([-4,1]) = [0,16]$;

b) the preimage of the set $[0,16]$ is the line segment $f^{-1}[0,16] = [-4,4]$. As we can see the preimage does not coincide with the given segment.

Example 2.4. Given a mapping $f: R \to R_+$, where $f(x) = |x|$ and two segments: $[-3,2]$ and $[-2,5]$. It is necessary to find images of these segments and preimages of these images.

The solution. The domain of this mapping is the set of nonnegative real numbers. The given mapping is also surjective. Let's find the image and its preimage for each of the segments:

a) $f([-3,2])=[0,3]$, $f^{-1}[0,3]=[-3,3]$;

b) $f([-2,5])=[0,5]$, $f^{-1}[0,3]=[-5,5]$.

It can be seen that the preimages also do not coincide with the given segments.

Example 2.5. Given a mapping $f: R\setminus\{0\}\to R$, where $f(x)=\ln|x|$ and two segments: $[1,e]$ and $[-e^2,-e]$. It is necessary to find images of these segments and pre preimages of these images.

The solution. The given mapping is also surjective. Let's find the image and its pre preimage for each of the two segments:

a) $f([1,e])=[0,1]$, $f^{-1}[0,1]=[-e,-1]\cup[1,e]$;

b) $f([-e^2,-e])=[1,2]$, $f^{-1}[1,2]=[-e^2,-e]\cup[e,e^2]$.

Example 2.6. Find the mapping of a segment $[0,1]$ into a segment $[a,b]$, where a,b are real numbers, $a<b$, which is realized by a linear function. Give a geometric interpretation of such a mapping.

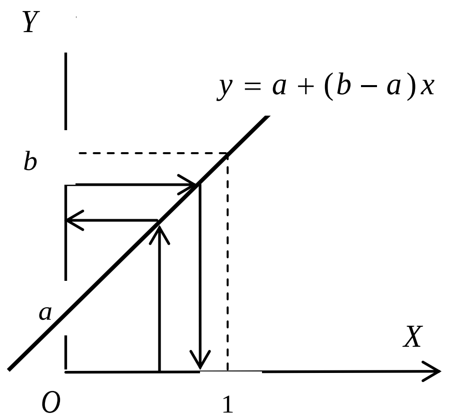

Fig. 2.5. A linear dependence

The solution. Let $x\in[0,1]$, and $y\in[a,b]$. Let's construct a linear relationship between these variables: $y=a+(b-a)x$. This is the desired mapping of a segment $[0,1]$ into a segment $[a,b]$. Now let's find out how this mapping is arranged from a geometric point of view. Let's plot the graph of the found dependence (Fig. 2.5).

Obviously, this mapping is a bijection: to any number $x\in[0,1]$ corresponds to a single number. And vice versa, any number corresponds to a single number $x\in[0,1]$. (Explain why?)

Thus, we showed that the sets $[0,1]$ and $[a,b]$ are of equipotent.

Example 2.7. The mapping $f: R\to R$ is given by the function $y=\sin x$. Find the image Z of the line segment $\left[0,\frac{\pi}{4}\right]$ and the preimage $f^{-1}(Z)$ of this image.

The solution. Let's write down the image of this segment: $f\left(\left[0,\frac{\pi}{4}\right]\right)=\left[0,\frac{\sqrt{2}}{2}\right]$. Now let's find out what will be the prototype of such an image. To do this, you need to plot the graph of the function $y=\sin x$, draw two horizontal lines $y=0$ and $y=\frac{\sqrt{2}}{2}$, project all the points of intersection of these lines with the graph of the function $y=\sin x$ on the axis OX. The union of an infinite number of received segments will represent the sought-after preimage $f^{-1}\left(\left[0,\frac{\sqrt{2}}{2}\right]\right)$. Draw two or three such segments on your own. What type of mapping is this?

So, finally we have: $f^{-1}\left(\left[0,\frac{\sqrt{2}}{2}\right]\right)=\left[2\pi n;\frac{\pi}{4}+2\pi n\right]\cup\left[\frac{3\pi}{4}+2\pi n;\pi+2\pi n\right]$, $n=0,\pm1,\pm2,\pm3,\cdots$.

Example 2.8. The mapping $f:\{1,2,3,4\}\to\{a,b,c,d\}$ is given by the following expression:

$$f(x)=\begin{cases} a, if\ \ x=1\ or\ x=3; \\ b, if\ \ x=2\ or\ x=4; \\ c, if\ \ x=2; \\ d, if\ \ x=3. \end{cases}$$

It is necessary:

a) to indicate the type of mapping;

b) find the range of values mapping;

c) find images of sets $\{1,4\}$, $\{2,3\}$, $\{1,3,4\}$;

d) find the preimages of the sets $f^{-1}\{a,b\}$, $f^{-1}\{c,d\}$, $f^{-1}\{b,d\}$, $f^{-1}\{b,c,d\}$.

The solution.

a) Since each element of the set $Y=\{a,b,c,d\}$ is the image of at least one element of the set $X=\{1,2,3,4\}$, then by definition we have a surrection.

b) The range of mapping values coincides with the set $Y=\{a,b,c,d\}$.

c) The image of the set $\{1,4\}$ consists of those elements of the set $Y=\{a,b,c,d\}$ that correspond to each of the elements 1 and 4. This is a set consisting of two elements, that is: $f\{1,4\}=\{a,b\}$. The image of the set $\{2,3\}$ consists of those elements of the set $Y=\{a,b,c,d\}$ that correspond to each of the elements 2 and 3. This set consists of four elements, that is: $f\{2,3\}=\{a,b,c,d\}$. For the latter, in this item of the task, we immediately write down the result of the reasoning: $f\{1,3,4\}=\{a,b,d\}$.

d) The preimage of a set $\{a,b\}$ is the set of all those elements x of the set $X=\{1,2,3,4\}$ for which the equality $f(x)=a$ or $f(x)=b$. So, we have: $f^{-1}\{a,b\}=\{1,2,3,4\}$. By similar considerations, we establish that $f^{-1}\{c,d\}=\{2,3\}$; $f^{-1}\{b,d\}=\{2,3,4\}$; $f^{-1}\{b,c,d\}=\{2,3,4\}$.

Tasks for classroom and independent work

2.1. Establish a one-to-one correspondence between the intervals $[0,1)$ and $[0,+\infty)$, that is, it is necessary to prove that these two sets are equipotent. Give an algebraic and geometric interpretation of such a mapping.

2.2. Establish a one-to-one correspondence between the intervals $(0,1)$ and $(-\infty,+\infty)$, that is, it is necessary to prove that these two sets are equipotent. Give an algebraic and geometric interpretation of such a mapping.

2.3. With the help of geometric images, establish a one-to-one correspondence between the set of points of the sphere and the plane, that is, it is necessary to prove that these two sets are equipotent.

I. Find the union $C = A \cup B$ and intersection $D = A \cap B$ of the two sets. Draw the obtained sets.

2.4. $A=\{(x,y)\,|\,x,y\in R, 1\le x\le 3; -2\le y\le 2\}$; $B=\{(x,y)\,|\,x,y\in R, 3x-2y=6\}$.

2.5. $A=\{(x,y)\,|\,x,y\in R, x^2+y^2<4\}$; $B=\{(x,y)\,|\,x,y\in R, x^2+y^2>1\}$.

2.6. $A=\{(x,y)\,|\,x,y\in R, x^2+y^2\le 36\}$; $B=\{(x,y)\,|\,x,y\in R, 2x-y\le 1\}$.

2.7. $A=\{(x,y)\,|\,x,y\in R, y-3x-8\le 0\}$; $B=\{(x,y)\,|\,x,y\in R, y\ge x^2/2\}$.

2.8. $A=\{(x,y)\,|\,x,y\in R, 1\le x\le 5; y\le x\}$; $B=\{(x,y)\,|\,x,y\in R, x^2+y^2\le 25\}$.

II. Find the intersection $D = A \cap B$ of two spatial sets. Draw the obtained sets.

2.9. $A=\{(x,y,z)\,|\,x,y,z\in R, x\ge 0; y\ge 0; z\ge 0\}$;

$B=\{(x,y,z)\,|\,x,y,z\in R, x+y+z\le 3\}$.

2.10. $A=\{(x,y,z)\,|\,x,y,z\in R, z\geq x^2+y^2\}$;

$B=\left\{(x,y,z)\,|\,x,y,z\in R, 8-z\leq\sqrt{x^2+y^2}\right\}$.

2.11. $A=\{(x,y,z)\,|\,x,y,z\in R, z\geq 2x^2+2y^2\}$;

$B=\left\{(x,y,z)\,|\,x,y,z\in R,\ z\leq 2\sqrt{x^2+y^2}\right\}$.

2.12. $A=\{(x,y,z)\,|\,x,y,z\in R, x^2+y^2+z^2\leq R^2\}$;

$B=\left\{(x,y,z)\,|\,x,y,z\in R, |z|\leq\sqrt{x^2+y^2}\right\}$.

2.13. Will an empty set be a set A whose elements are empty sets, i.e. $A=\{\varnothing,\varnothing,\varnothing,\cdots,\varnothing\}$?

2.14. What set will be the intersection $P=\bigcap_{i=1}^{\infty}A_i$ of the counted family of sets (intervals) A_n if $A_n=\left(\frac{n-1}{2n+1};\frac{n+1}{2n-1}\right)$, $n\in N$?

2.15. Find the intersection $P=\left(\bigcap_{i=1}^{\infty}A_i\right)\bigcap\left(\bigcap_{i=1}^{\infty}B_i\right)$ of two intersections of infinite families of sets $\{A_n\}$ and $\{B_n\}$, respectively, if $A_n=\left(\frac{2n-1}{3n+1};\frac{5n+1}{6n-1}\right)$, $B_n=\left(\frac{3n-1}{4n+1};\frac{9n+1}{10n-1}\right)$, $n\in N$.

2.16. Show that the set of natural numbers $N=\{1,2,3,\cdots,n,n+1,\cdots\}$ and the set of integers $Z=\{0,+1,-1,+2,-2,\cdots,+M,-M,\cdots\}$ are equipotent (or equivalent). Construct the appropriate mapping in the form of a function, which connects the elements of these two sets.

2.17. Define the power of a set. Describe the power of the counted set. How is it indicated?

2.18. Describe the power of uncountable sets. What sets of the continuum type do you know? What are their properties? Are there sets whose capacity is greater than $\aleph_1$?

2.19. Prove that the set of all rational numbers is countable.

2.20. The mapping $f:\left[0,\frac{\pi}{2}\right]\to[0,1]$ is given by the function $y=\sin^2 x$. It is necessary to find out what kind of mapping this is – surjective, injective or bijective? Is there an inverse function for this function?

2.21. Let two sets be given: $N = \{1, 2, 3, \cdots, n, n+1, \cdots\}$ and $B = \{0, 1\}$. The following correspondence is established between the elements of these sets $f : N \to B$:

$$y = f(x) = \frac{1-(-1)^x}{2} = \begin{cases} 0, & if\ x\ is\ an\ even\ number; \\ 1, & if\ x\ is\ an\ odd\ number. \end{cases}$$

It is necessary to find out what kind of mapping this is – surjective, injective or bijective? Is there an inverse function for this function?

Answers and instructions

2.1. For example, the function $y = \operatorname{tg}\left(\frac{\pi}{2}x\right) : [0, 1) \to [0, +\infty)$ (Fig. 2.6).

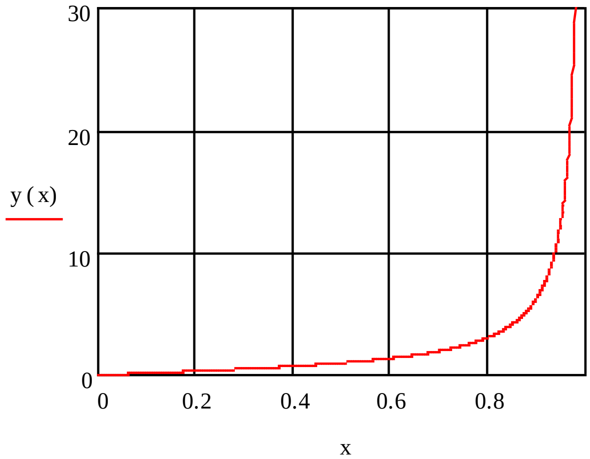

Fig. 2.6

2.2. For example, the function $y = \ln\left(\frac{x}{1-x}\right) : (0, 1) \to (-\infty, +\infty)$ (Fig. 2.7).

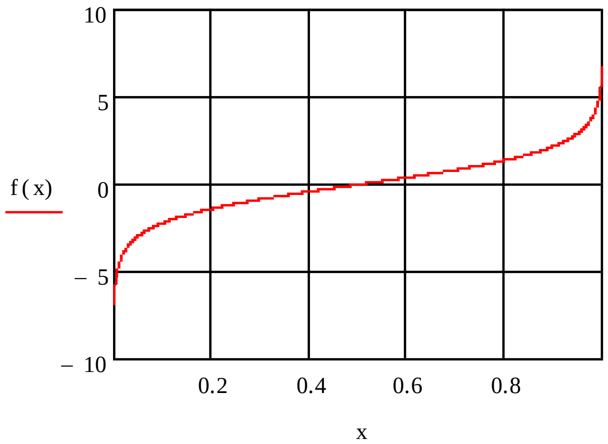

Fig. 2.7

2.3. *Instruction.* Place the sphere S so (Fig. 2.8) that it touches the OXY plane at the origin of coordinates O. Sphere S is intersected by straight lines emanating from the same center – pole P. From this figure, it will be clear how the mutually unambiguous mapping of the sphere onto plane OXY is arranged. Points $\{N_i\}$ mark the intersection of these straight lines with sphere S, and points $\{M_i\}$ – their intersection with plane OXY. Each point N_i on the sphere S corresponds to its own point M_i on the plane OXY, and vice versa. The only exception is the P pole, which corresponds to infinitely distant points on the OXY plane. At this point, the mutually unambiguous mapping of the sphere onto the OXY plane is broken.

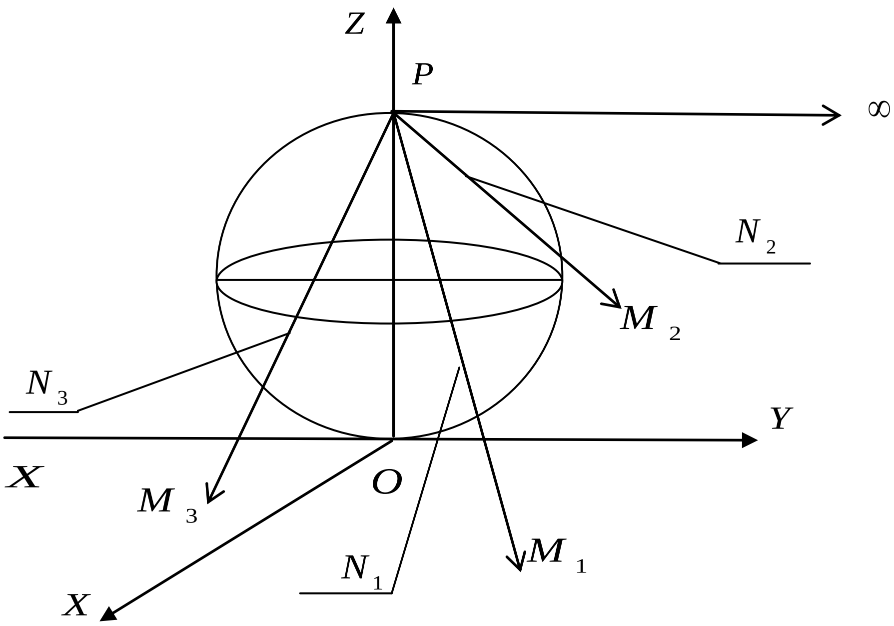

Fig. 2.8

2.13. No, not empty.

2.14. $P = \{1/2\}$.

2.15. $P = \{x \mid x \in [3/4;\ 5/6]\}$.

2.16. $f(x) = \begin{cases} +k, if \ \ x = 2k+1, \ (k = 0,1,2,3,\cdots); \\ -m, if \ \ x = 2m, \ (1,2,3,\cdots). \end{cases}$

2.20. A bijective. The inverse function exists: $x = \arcsin\sqrt{y}$.

2.21. A surjective, inverse function does not exist.

Topic II. The limit of a numerical sequence and the limit of a function

PRACTICAL LESSON 3
Numerical sequences and their limit. Justification of the convergence of numerical sequences. Recurrent numerical sequences. Stolz's theorem

3.1. Definition of a numerical sequence and its limit. The number e. Theorem about "two policemen"

Definition 3.1. *A* ***numerical sequence*** *is called a function* $f : N \to R$*, which matches each natural number* $n \in N$ *with some real number* $x_n \in R$*, i.e.* $x_n = f(n)$. Therefore, a numerical sequence is a variable with numbered values. The sequence is denoted by letters with indices: $x_1, x_2, x_3, \cdots, x_n, \cdots$, or $\{x_n\}$ in short, where x_n is a common member of the sequence.

Definition 3.2. *A sequence* $\{x_n\}$ *is called* ***convergent*** *if there exists a number* x_0 *such that for an arbitrary number* $\varepsilon > 0$ *there exists a number* $N = N(\varepsilon)$ *such that for all* $n > N(\varepsilon)$ *the inequality holds*

$$\left| x_n - x_0 \right| < \varepsilon .$$

In this case, *the number* x_0 *is called the* ***limit of the sequence*** $\{x_n\}$. Convergence of the sequence $\{x_n\}$ to its limit (numbers x_0) is concisely denoted as follows:

$$\lim_{n \to \infty} x_n = x_0 ,$$

or $x_n \to x_0$ by $n \to \infty$.

Geometrically, the definition of the boundary of a numerical sequence means that starting from the number $n > N(\varepsilon)$, all members of the sequence $\{x_n\}$ will belong to the interval $(x - \varepsilon, x + \varepsilon)$, which is called the ε-***circle*** of the boundary point x_0.

Number e. We present the formula for finding the ***second important limit*** of a special numerical sequence $\{x_n\}$, which generates the transcendental number e:

$$\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e . \qquad (3.1)$$

Formula (3.1) will later play an important role in the calculations of various limits. In addition, the results of such a theorem are often used in some theoretical and practical problems.

Theorem ("*about two policemen*"). *If the numerical sequences* $\{x_n\}$ *and* $\{z_n\}$ *are* ***convergent****, have a common boundary* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, *and* $x_n \le y_n \le z_n$ *by* $n \ge N_0$, *then the sequence* $\{y_n\}$ *is also* ***convergent****; at the same time, the boundaries of all three sequences coincide:*

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n .$$

Let's consider a number of examples of finding the limit of various numerical sequences.

Example 3.1. Find the limit of the numerical sequence $\{x_n\}$: $x_n = \sqrt[n]{n}$.

The solution. To calculate this limit, we use the well-known inequality discussed above:

$$n^{n+1} > (n+1)^n, \quad n \ge 3 .$$

It follows that $n > \left(1 + \frac{1}{n}\right)^n$. The plan of the following transformations is to construct a series of inequalities, which is used in the "two policemen" theorem.

Let's do some transformations. Let's denote $a = 1 + \lambda$ and expand the expression $a^n = (1+\lambda)^n$ according to Newton's binomial formula:

$$a^n = (1+\lambda)^n = 1 + n\lambda + \frac{n(n-1)}{2!}\lambda^2 + \cdots + \lambda^n > \frac{n(n-1)}{2!}\lambda^2, \quad \lambda = a - 1 .$$

It is obvious that the inequality $n - 1 > \frac{n}{2}$ holds for $n > 2$, therefore

$$a^n > \frac{n^2 (a-1)^2}{4} .$$

We replace $a = \sqrt[n]{n}$ and write the previous inequality in the new variables:

$$n > \frac{n^2 (\sqrt[n]{n} - 1)^2}{4} \;\Rightarrow\; \sqrt[n]{n} - 1 < \frac{2}{\sqrt{n}} \;\Rightarrow\; \sqrt[n]{n} < 1 + \frac{2}{\sqrt{n}} .$$

The first inequality is obtained – the restriction of the $\sqrt[n]{n}$ expression from above. Now we obtain the second inequality to limit the same expression from below. Let's use the inequality $n > \left(1 + \frac{1}{n}\right)^n$ obtained at the beginning of the solution, from which we have the estimate $\sqrt[n]{n} > 1 + \frac{1}{n}$. Now let's write the double inequality:

$$1 + \frac{1}{n} < \sqrt[n]{n} < 1 + \frac{2}{\sqrt{n}}.$$

As a result of the transition to the limit for $n \to \infty$ in these inequalities, we obtain the desired limit:

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Note that there is another way of calculating this limit.

Example 3.2. Find the limit of the numerical sequence $\{x_n\}$: $x_n = \sqrt[n]{a}$, $a > 1$.

The solution. To calculate this limit, we use Bernoulli's inequality:

$$\gamma^n > 1 + n(\gamma - 1), \ \forall \gamma > 1, \ n \ge 2.$$

It follows from the well-known inequality $(1+\alpha)^n > 1 + n\alpha$ if $\gamma = \alpha + 1$ is taken.

So, let's make the $\gamma = \sqrt[n]{a}$ replacement and write the corresponding Bernoulli inequality in new notation:

$$(\sqrt[n]{a})^n > 1 + n(\sqrt[n]{a} - 1) \Rightarrow a > 1 + n(\sqrt[n]{a} - 1) \Rightarrow \sqrt[n]{a} - 1 < \frac{a-1}{n}.$$

An upper bound for the expression $(\sqrt[n]{a} - 1)$ is obtained. Since $a > 1$, the value of $(\sqrt[n]{a} - 1)$ is bounded from below by zero; we have the following double inequality:

$$0 < \sqrt[n]{a} - 1 < \frac{a-1}{n}, \ a > 1.$$

In these inequalities, let's go to the limit for $n \to \infty$. According to the theorem "about two policemen", we obtain the desired limit of the sequence $x_n = \sqrt[n]{a}$:

$$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

Note that this limit can be calculated in another way.

Example 3.3. Find the limit $\lim_{n\to\infty} x_n$ of a numerical sequence

$$\{x_n\} = \left\{ \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} \right\}.$$

The solution. To calculate this limit, we take out the largest terms in the numerator and denominator in parentheses and reduce by 3^n:

$$x_n = \frac{1}{3} \cdot \frac{[(-2/3)^n + 1]}{[(-2/3)^{n+1} + 1]}.$$

After passing to the border, we get:

$$\lim_{n\to\infty} x_n = \frac{1}{3} \lim_{n\to\infty} \frac{[(-2/3)^n + 1]}{[(-2/3)^{n+1} + 1]} = \frac{1}{3}.$$

Example 3.4. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = \frac{1 + a + a^2 + \cdots + a^n}{1 + b + b^2 + \cdots + b^n}, \; |a| < 1, \; |b| < 1.$$

The solution. We have geometric progressions in the numerator (factor a) and in the denominator (factor b). To calculate the limits of the sequence, we write down the sums of these geometric progressions:

$$x_n = \frac{\dfrac{1 - a^{n+1}}{1 - a}}{\dfrac{1 - b^{n+1}}{1 - b}} = \frac{1 - b}{1 - a} \cdot \frac{1 - a^{n+1}}{1 - b^{n+1}}.$$

Since $a^{n+1} \to 0$ and $b^{n+1} \to 0$ are given by $n \to \infty$, after passing to the boundary we obtain:

$$\lim_{n\to\infty} x_n = \frac{1 - b}{1 - a} \lim_{n\to\infty} \frac{1 - a^{n+1}}{1 - b^{n+1}} = \frac{1 - b}{1 - a}.$$

Example 3.5. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = \sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} \cdot \cdots \cdot \sqrt[2^n]{2}.$$

The solution. Let's make elementary transformations:

$$x_n = \sqrt{2} \cdot \sqrt[4]{2} \cdot \sqrt[8]{2} \cdot \cdots \cdot \sqrt[2^n]{2} = 2^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}} = 2^{1 - \frac{1}{2^n}}.$$

We find the limit of the resulting expression:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 2^{1 - \frac{1}{2^n}} = \left| \frac{1}{2^n} \to 0 \right| = 2.$$

Example 3.6. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

The solution. We will use two inequalities that were considered during the study of the topic "Mathematical Induction Method":

$$\left(1+\frac{1}{n}\right)^n < e \quad \text{and} \quad \left(1+\frac{1}{n}\right)^{n+1} > e, \quad n \geq 1.$$

We prologarithmize the first of these two inequalities:

$$n\ln\left(1+\frac{1}{n}\right) < 1 \Rightarrow \ln\left(1+\frac{1}{n}\right) < \frac{1}{n} \Rightarrow \ln\left(1+n\right) - \ln n < \frac{1}{n}.$$

We will successively give the natural number n increasing values (starting with $n+1$) and write the resulting inequalities one below the other in the following way:

$$\ln\left(n+2\right) - \ln(n+1) < \frac{1}{n+1};$$

$$\ln\left(n+3\right) - \ln(n+2) < \frac{1}{n+2};$$

$$\ln\left(n+4\right) - \ln(n+3) < \frac{1}{n+3};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\ln\left(2n+1\right) - \ln(2n) < \frac{1}{2n}.$$

Now add all these inequalities; as a result, we get the following inequality:

$$\ln\left(2n+1\right) - \ln(n+1) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \Rightarrow \ln\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

We prologarithmize the second of the mentioned two inequalities:

$$(n+1)\ln\left(1+\frac{1}{n}\right) > 1 \Rightarrow \ln\left(1+\frac{1}{n}\right) > \frac{1}{n+1} \Rightarrow \ln\left(1+n\right) - \ln n > \frac{1}{n+1}.$$

Next, similarly to the previous one, we obtain the following inequality:

$$\ln\left(2n\right) - \ln(n) > \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \Rightarrow \ln 2 > \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}.$$

Let's combine both obtained inequalities into one dual:

$$\ln\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \ln 2 .$$

Let's pass in it to the boundary for $n \to \infty$; according to the theorem "about two policemen" we have:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right) = \ln 2 . \qquad (3.2)$$

We get an interesting and useful result: $\lim_{n\to\infty} x_n = \ln 2$.

Formula (3.1) is often used in practical work with numerical series.

Remarks 3.1. In addition, one more useful inequality of the following form was obtained:

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n},$$

used when proving the existence of the number e.

Remarks 3.2. From the last inequality, substituting the variables, you can get the following inequality:

$$\ln(1+\alpha) < \alpha \Rightarrow e^{\alpha} > 1 + \alpha , \; \alpha > 0 .$$

Example 3.7. Find the limit $\lim_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = n(\sqrt[n]{a} - 1) , \; a > 0 .$$

The solution. We introduce the substitution $z = \frac{1}{n}$ and rewrite the general term of the sequence as follows:

$$n(\sqrt[n]{a} - 1) = \frac{a^{\frac{1}{n}} - 1}{\frac{1}{n}} = \frac{a^z - 1}{z} , \; z = \frac{1}{n} \to 0 , \text{ if } n \to \infty .$$

Now the task is reduced to calculating the following limit:

$$\lim_{z\to 0} \frac{a^z - 1}{z} = \left| \begin{array}{l} y = a^z - 1; y \to 0 \\ z = \log_a(y+1); \end{array} \right| = \lim_{y\to 0} \frac{y}{\log_a(y+1)} = \lim_{y\to 0} \frac{1}{\frac{\log_a(y+1)}{y}} =$$

$$= \lim_{y\to 0} \frac{1}{\log_a(y+1)^{\frac{1}{y}}} = \left| \lim_{y\to 0}(y+1)^{\frac{1}{y}} = e \right| = \frac{1}{\log_a e} = \ln a .$$

The process of solving this problem will be significantly reduced if you use the equivalence table. For the function $\sqrt[n]{a}-1$, this equivalence has the form

$$\sqrt[n]{a}-1=e^{\frac{1}{n}\ln a}-1\sim\left|\begin{array}{l} e^{z}\sim 1+z+\dfrac{z^{2}}{2!}+\cdots, \\ if\ \ z\to 0,\ z=\dfrac{1}{n}\ln a \end{array}\right|\sim\frac{1}{n}\ln a+\frac{1}{2n^{2}}\ln^{2}a+\cdots,\ \text{if}\ \ n\to\infty .$$

Further, this problem is easy to solve.

Therefore, the limit of the given sequence is as follows: $\lim\limits_{n\to\infty}x_n=\ln a$.

Example 3.8. Find the limit $\lim\limits_{n\to\infty}x_n$ of a numerical sequence

$$\{x_n\}=\left\{n^2(\sqrt[n]{a}-\sqrt[n+p]{a})\right\}$$

where $a>0$, $p>0$ is a fixed real number.

The solution. We use the results of the previous problem and write down the expression of the general term x_n of the sequence using the equivalences given above ($n\to\infty$):

$$\sqrt[n]{a}=e^{\frac{1}{n}\ln a}\approx 1+\frac{1}{n}\ln a\,;$$

$$\sqrt[n+p]{a}=e^{\frac{1}{n+p}\ln a}\approx 1+\frac{1}{n+p}\ln a\,;$$

$$x_n=n^2(\sqrt[n]{a}-\sqrt[n+p]{a})\approx n^2\left(1+\frac{1}{n}\ln a-1-\frac{1}{n+p}\ln a\right)=\frac{np}{n+p}\ln a\,.$$

We will have a limit in the resulting expression for $n\to\infty$:

$$\lim_{n\to\infty}x_n=p\cdot\ln a\cdot\lim_{n\to\infty}\frac{n}{n+p}=p\cdot\ln a\,.$$

3.2. The technique of calculating the limits of recurrent numerical sequences

Consider a typical example of calculating the limit of a recurrent numerical sequence without proving its convergence. The approach used in this example belongs to Euler.

Example 3.9 (*recurrent sequences*). Find the limit $\lim\limits_{n\to\infty}x_n$ of the numerical sequence $\{x_n\}$, which is given by the recurrent formula:

$$x_{n+2} = \frac{x_{n+1} + x_n}{2}, \; x_1 = a, \; x_2 = b.$$

The solution. Let's rewrite the given recurrence relation as follows: $2x_{n+2} - x_{n+1} - x_n = 0$. A second-order linear equation in finite differences is obtained, the solution of which is sought using the following substitution (Euler): $x_n = q^n$, where q is an unknown number. We will find this number. by substituting this expression into the equation in finite differences:

$$2q^2 - q - 1 = 0 \Rightarrow q_1 = 1, q_2 = -\frac{1}{2}.$$

We build a junction according to the following scheme:

$$x_n = C_1 q_1^n + C_2 q_2^n, \tag{3.3}$$

where C_1, C_2 are unknown constants. We determine them from the initial conditions of the problem $x_2 = b$ $x_1 = a$:

$$x_1 = a = C_1 q_1 + C_2 q_2 = C_1 - \frac{1}{2} C_2;$$

$$x_2 = b = C_1 q_1^2 + C_2 q_2^2 = C_1 + \frac{1}{4} C_2.$$

A system of two linear equations with respect to unknown C_1, C_2 constants is obtained:

$$\begin{cases} C_1 - \frac{1}{2} C_2 = a; \\ C_1 + \frac{1}{4} C_2 = b. \end{cases}$$

From this system we find:

$$C_1 = \frac{a + 2b}{3}, C_2 = \frac{4}{3}(b - a).$$

Let's substitute the obtained constants into formula (3.3) and construct an expression for the required limit:

$$x_n = \frac{a + 2b}{3} \cdot 1^n + \frac{4}{3}(b - a)\left(-\frac{1}{2}\right)^n.$$

Now we will find the limit of this expression for $n \to \infty$:

$$\lim_{n \to \infty} x_n = \frac{a + 2b}{3}.$$

We will meet with the solution of other examples on the topic of calculating the limits of recurrent sequences in the next paragraph. But before calculating the limits of such sequences, the proof of their existence will be given using an important theorem on the necessary and sufficient conditions for the convergence of numerical sequences.

3.3. The technique of applying the theorem on necessary and sufficient conditions for the convergence of a numerical sequence. Use of Stolz's theorem to find limits of sequences

In the future, we will use the important ***theorem on the necessary and sufficient conditions*** for the existence of the limit of a numerical sequence.

Theorem. *If a **nondecreasing (nonincreasing)** sequence* $\{x_n\}$ *is bounded **from above (from below)**, then it is **convergent**.*

Therefore, to prove that a given sequence $\{x_n\}$ is convergent, two facts must be established:

a) ***limitation*** of sequence $\{x_n\}$;

b) its ***monotony***.

We will show how this theorem "works" in practice.

Example 3.10. Prove that the sequence $\{x_n\}$: $x_n = \frac{10}{1} \cdot \frac{11}{3} \cdot \frac{12}{5} \cdot \ldots \cdot \frac{n+9}{2n-1}$ is convergent.

The solution. We immediately note that for $n = 10$, the fractional factor $q(n) = \frac{n+9}{2n-1}$, which forms the common term of the sequence x_n, becomes equal to one and then decreases to $\frac{1}{2}$ as the number n increases, that is, $q(n) < 1$ for $n > 10$. To investigate the monotonicity of $\{x_n\}$, consider the proportion of the ratio of two consecutive elements of this sequence:

$$\frac{x_{n+1}}{x_n} = \frac{n+10}{2n+1} = q(n+1) < 1 \;\Rightarrow\; x_{n+1} < x_n,\; n > 9\,.$$

Hence, the sequence $\{x_n\}$ is monotonically decreasing. Now let's check its boundedness from below. Since all factors $q(n)$ that form the common member of the sequence x_n are positive, their product is always greater than zero, that is, all elements of $\{x_n\}$ are bounded ***from below*** by zero.

Thus, it is proved that the given sequence is monotonically decreasing and bounded from below. Therefore, according to the given theorem, it is ***convergent***. It is obvious that the limit of this sequence is zero.

Example 3.11. Prove that the sequence $\{x_n\}$: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right), x_1 > 0$ is convergent.

The solution. First, we prove that the sequence is bounded from below:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right); t = x_n \Rightarrow t + \frac{1}{t} \ge 2 \Rightarrow x_n \ge 1,$$

if $n \ge 1$.

Thus, the sequence $\{x_n\}$ is bounded from below. Now let's show it monotony. To do this, we construct and estimate the value of the ratio of two adjacent elements $\{x_n\}$:

$$\frac{x_{n+1}}{x_n} = \frac{1}{2}\left(1 + \frac{1}{x_n^2}\right) \le 1 \Rightarrow x_{n+1} \le x_n .$$

Therefore, the sequence is ***nonincreasing*** and is ***bounded from below***, so it has a certain limit. Let us denote this limit by a and find it. To do this, let's go to the limit in the recurrent formula that specifies the sequence $\{x_n\}$:

$$\lim_{n\to\infty}(x_{n+1}) = \lim_{n\to\infty}\frac{1}{2}\left(x_n + \frac{1}{x_n}\right) \Rightarrow a = \frac{1}{2}\left(a + \frac{1}{a}\right) \Rightarrow a^2 - 1 = 0 .$$

The only root that satisfies the condition of the problem is $a = 1$.

How can the convergence process of this sequence be graphically interpreted?

Remark. However, there are sequences ***that are not monotonic, but convergent***.

Stolz's theorem is used to find the limits of some sequences.

Theorem (*Stolz*). *Let $\{y_n\}$ be an infinitely large increasing sequence and let the sequence $\left\{\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right\}$ be convergent and have the limit a. Then the sequence $\{x_n/y_n\}$ is also convergent and has the limit a. Therefore, the following equality holds:*

$$\lim_{n\to\infty}\left\{\frac{x_n}{y_n}\right\} = \lim_{n\to\infty}\left\{\frac{x_n - x_{n-1}}{y_n - y_{n-1}}\right\}.$$

We will give several examples using this theorem.

Example 3.12. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = \frac{\ln\left[\left(2+\frac{1}{2}\right)\left(2^2+\frac{1}{2^2}\right)\left(2^3+\frac{1}{2^3}\right)\cdots\left(2^n+\frac{1}{2^n}\right)\right]}{2n^2}.$$

The solution. We have the quotient of the common terms of two infinitely large sequences $x_n = \frac{u_n}{v_n}$. Let's use Stolz's theorem:

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\left(\frac{u_n - u_{n-1}}{v_n - v_{n-1}}\right) = \lim_{n\to\infty}\left(\frac{\ln\left(2^n+\frac{1}{2^n}\right)}{2n^2-2(n-1)^2}\right) = \lim_{n\to\infty}\left(\frac{n\ln 2}{4n}\right) = \frac{\ln 2}{4}.$$

Example 3.13. Find the limit $\lim\limits_{n\to\infty} x_n$ of a numerical sequence

$$\{x_n\} = \left\{\frac{n}{\sqrt[n]{n!}}\right\}.$$

The solution. Let us present the general term of the sequence as follows:

$$x_n = \frac{n}{\sqrt[n]{n!}} = e^{\ln\left(\frac{n}{\sqrt[n]{n!}}\right)} = e^{\ln n - \frac{1}{n}(\ln 1 + \ln 2 + \cdots + \ln n)}.$$

Let's consider the following sequence separately:

$$z_n = \ln n - \frac{1}{n}(\ln 1 + \ln 2 + \cdots + \ln n).$$

Let's use Stolz's theorem and find its limit:

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}\left(\ln n - \frac{1}{n}(\ln 1 + \ln 2 + \cdots + \ln n)\right) = \lim_{n\to\infty}\left[\frac{n\ln n - (\ln 1 + \ln 2 + \cdots + \ln n)}{n}\right] =$$

$$= \left|\lim_{n\to\infty}\left\{\frac{u_n}{v_n}\right\} = \lim_{n\to\infty}\left\{\frac{u_n - u_{n-1}}{v_n - v_{n-1}}\right\}\right| = \lim_{n\to\infty}\left[\frac{n\ln n - (n-1)\ln(n-1) - \ln n}{1}\right] =$$

$$= \lim_{n\to\infty}\left[(n-1)\ln\left(\frac{n}{n-1}\right)\right] = \lim_{n\to\infty}\left[\ln\left(1+\frac{1}{n-1}\right)^{(n-1)}\right] = \ln e = 1.$$

Now let's write the sought limit $\lim\limits_{n\to\infty} x_n$:

$$\lim_{n\to\infty} x_n = e^1 = e.$$

So, it is proved:

$$\lim_{n\to\infty}\left(\frac{n}{\sqrt[n]{n!}}\right)=e\,.$$

Example 3.14. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n=\frac{1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n(n+1)(n+2)}{n^4+n^2+7}\,.$$

The solution. We have the quotient of the common terms of two infinitely large sequences u_n and v_n: $x_n=\dfrac{u_n}{v_n}$. Let's use Stolz's theorem; get:

$$\lim_{n\to\infty} x_n=\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right)=\lim_{n\to\infty}\left(\frac{u_n-u_{n-1}}{v_n-v_{n-1}}\right)=\lim_{n\to\infty}\left(\frac{n(n+1)(n+2)}{n^4+n^2+7-\left((n-1)^4+(n-1)^2+7\right)}\right)=$$

$$=\lim_{n\to\infty}\left(\frac{n(n+1)(n+2)}{4n^3+\cdots}\right)=\frac{1}{4}\,.$$

Here, the terms that have a degree less than 3 are marked with dots (***the main term*** is highlighted).

Example 3.15. Find the limit $\lim\limits_{n\to\infty} z_n$ of the numerical sequence $\{z_n\}$ with complex terms:

$$z_n=\frac{(n+5)\cdot 5^n+n^5\cdot 4^n i}{(n+1)3^n+n^3\cdot(-2)^n i}\,.$$

The solution. Let's take out the corresponding factors in parentheses that determine the behavior of the numerator and denominator at infinity. After that, we go to the boundary:

$$\lim_{n\to\infty} z_n=\lim_{n\to\infty}\frac{(n^3+5)\cdot 5^n+n^5\cdot 4^n i}{(n^2+1)(-3)^n+n^3\cdot 5^n i}=\lim_{n\to\infty}\frac{5^n\left[(n^3+5)+n^5\cdot\left(\frac{4}{5}\right)^n i\right]}{\left[(n^2+1)\frac{(-3)^n}{5^n}+n^3\cdot i\right]5^n}=$$

$$=\left|\begin{array}{l} n^5\cdot\left(\frac{4}{5}\right)^n\to 0,\, if\;\; n\to\infty; \\ (n^2+1)\frac{(-3)^n}{5^n}\to 0,\, if\;\; n\to\infty \end{array}\right|=\lim_{n\to\infty}\frac{n^3+5}{n^3\cdot i}=-i\,.$$

Tasks for classroom and independent work

3.1. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = n^3\left(\sqrt[n]{a} - \sqrt[n+p]{a} - \frac{p}{n(n+p)}\ln a\right),$$

$a > 0$, $p > 0$ are fixed real numbers.

3.2. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = \left(\frac{\sqrt[n]{a} + \sqrt[n]{b}}{2}\right)^n, |a| > 1, |b| > 1 \text{ are fixed real numbers.}$$

3.3. Prove that the given sequence $\{x_n\}$ is convergent:

$$x_n = \left(1 - \frac{1}{2}\right)\cdot\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2^n}\right).$$

3.4. Prove that the given sequence $\{x_n\}$ is convergent:

$$x_n = \left(1 + \frac{1}{2}\right)\cdot\left(1 + \frac{1}{4}\right)\cdots\left(1 + \frac{1}{2^n}\right).$$

3.5. Prove that a recursively given sequence $\{x_n\}$ is convergent and find its limit: $x_{n+1} = x_n(2 - 6x_n), 0 < x_1 < \dfrac{1}{6}$. Draw a diagram of the sequence convergence process.

3.6. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$, which is given by the recurrent formula: $x_{n+2} = \dfrac{3x_{n+1} - x_n}{2}$, $x_1 = a$, $x_2 = b$.

3.7. Prove that a recursively given sequence $\{x_n\}$ is convergent and find its limit: $x_{n+1} = \dfrac{1}{3}(1 + \dfrac{1}{x_n^2})$, $x_1 = 3$. Draw a diagram of the sequence convergence process.

3.8. Find the limit $\lim\limits_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = \frac{1^{100} + 2^{100} + 3^{100} + \cdots + n^{100}}{n^{100}} - \frac{n}{101}.$$

3.9. Prove that the given equality is correct:

$$\lim_{n\to\infty}\sqrt[n]{x_1\cdot x_2\cdot x_3\cdot\cdots\cdot x_n} = \lim_{n\to\infty} x_n, \text{ where } \forall n : x_n > 0.$$

3.10. Find the limit $\lim_{n\to\infty} y_n$ of the numerical sequence $\{y_n\}$:

$$y_n = \frac{1^2 \cdot x_1^2 + 2^2 \cdot x_2^2 + 3^2 \cdot x_3^2 + \cdots + n^2 \cdot x_n^2}{2n^3 + 3n^2 + 2n + 5}, \text{ if } \lim_{n\to\infty} x_n = x_0 .$$

3.11. Find the limit $\lim_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = \frac{\ln\left[\left(e^p + \frac{1}{2}\right)\left(e^p + \frac{1}{3}\right)\left(e^p + \frac{1}{4}\right)\cdots\left(e^p + \frac{1}{n}\right)\right]}{pn}.$$

3.12. Find the limit $\lim_{n\to\infty} x_n$ of the numerical sequence $\{x_n\}$:

$$x_n = \frac{1^{1000} + 2^{1000} + 3^{1000} + \cdots + n^{1000}}{n^{1001}}.$$

3.13. Find the limit $\lim_{n\to\infty} z_n$ of the numerical sequence $\{z_n\}$ with complex terms:

$$z_n = \frac{(-3)^n + n \cdot 4^n i}{n \cdot 2^{2n} - (-4)^n i}.$$

3.14. Give an example of a convergent sequence that is not monotonic.

Answers and instructions

3.1. $p \ln^2 a$. **3.2.** $\sqrt{ab}$. **3.3.** Convergent.

3.4. Convergent. **3.5.** $1/6$. **3.6.** $2b - a$.

3.8. $1/2$. **3.10.** $x_0^2/6$. **3.11.** 1.

3.12. $\frac{1}{1001}$ **3.13.** i.

PRACTICAL LESSON 4
The limit of a function of one variable. Cauchy criterion. Infinitely large and infinitely small functions. Equivalent infinitesimals. Table of equivalences. The technique of revealing the main uncertainties

Remark. Two practical classes should be taken to master the material presented here. At the beginning of this topic, we will present the main concepts and definitions that will be used further.

4.1. Two definitions of the limit of a function (according to Heine and according to Cauchy). The Cauchy criterion for the limit of a function

First Definition of the limit of a function (Heine). The number **A** is called the **limit of the function** $y = f(x)$ **at the point** x_0, if for an arbitrary sequence $\{x_n\}$ converging to x_0, where $x_n \in X$, $x_n \neq x_0$, the sequence of values of the function $\{f(x_n)\}$ has a limit equal to the number **A.**

Second Definition of the limit of a function (Cauchy). A number **A** is called the limit of a function $y = f(x)$ at the point x_0, if for an arbitrary number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that for all $x \in X$ that satisfy the inequality $0 < |x - x_0| < \delta$, the inequality $|f(x) - A| < \varepsilon$ is satisfied.

Definition. A number **A** is called **the limit of a function** $y = f(x)$ **on the left** (or the left limit) at the point x_0, if for any number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that for all $x \in (x_0 - \delta; x_0)$ the inequality $|f(x) - A| < \varepsilon$ is fulfilled.

Definition. A number B is called the **limit of a function** $y = f(x)$ **on the right** (or the right limit) at the point x_0, if for any number $\varepsilon > 0$ there exists a number $\delta = \delta(\varepsilon) > 0$ such that when $x \in (x_0; x_0 + \delta)$ the inequality $|f(x) - B| < \varepsilon$ is satisfied.

Theorem 4.1. *Let the function* $y = f(x)$ *be defined in some neighborhood* X *of the point* x_0 *(except, perhaps, the point* x_0 *itself). Then, in order for the* *function* $y = f(x)$ *to have a limit at the point* x_0*, it is necessary and sufficient for its right* $\lim\limits_{x \to x_0+0} f(x)$ *and left* $\lim\limits_{x \to x_0-0} f(x)$ *limits to exist and these limits to coincide, i.e.*

$$\left(\lim_{x \to x_0+0} f(x) = \lim_{x \to x_0-0} f(x) = A\right) \Leftrightarrow \lim_{x \to x_0} f(x) = A.$$

Theorem 4.2 (Cauchy Criterion). In order for there to be a limit of the function $f(x)$ at $x \to x_0$, it is necessary and sufficient for an arbitrary number $\varepsilon > 0$ to find a number $\delta = \delta(\varepsilon) > 0$ such that for any arguments $x, x' \in X$ that satisfy the inequalities $0 < |x - x_0| < \delta(\varepsilon)$ and $0 < |x' - x_0| < \delta(\varepsilon)$, the following inequality would hold for the difference of the corresponding values of the function: $|f(x) - f(x')| < \varepsilon$.

4.2. Definition of infinitely *small* and infinitely large functions. The first and second important limits, the technique of their calculation

Definition. A function $y = f(x)$ at $x \to x_0$ is called **infinitely large** (that is, has a limit ∞) if it is defined in some neighborhood of the point x_0, except, perhaps, the point x_0 itself, and for an arbitrary number $M > 0$ there is a number $\delta = \delta(M) > 0$ such that for all x, which satisfy the inequalities $0 < |x - x_0| < \delta$, the inequality $|f(x)| > M$ is fulfilled. It is denoted as follows: $\lim\limits_{x \to x_0} f(x) = \infty$ or $f(x) \to \infty$ at $x \to x_0$.

Definition. An **infinitely small value** around a point x_0 is a variable value (function) whose limit at this point is **equal to zero**.

In particular, a function $\alpha(x)$ is called an **infinitesimal value** (or an **infinitesimal function**) if $x \to x_0$ or $x \to \infty$ if : $\lim\limits_{x \to x_0} \alpha(x) = 0$; $\lim\limits_{x \to \infty} \alpha(x) = 0$.

We will give the definition of an **infinitesimally small value** in the language "$\varepsilon - \delta$".

Definition. A function $\alpha(x)$ is called **infinitesimally small value** at $x \to x_0$ ($x \to \infty$) if for an arbitrary number $\varepsilon > 0$ there exists such a number $\delta = \delta(\varepsilon) > 0$ $(M > 0)$ that for all x satisfying the inequality $|x - x_0| < \delta$ $(|x| > M)$, the inequality $|\alpha(x)| < \varepsilon$ is fulfilled.

To reveal the classical uncertainty $\left\{\frac{0}{0}\right\}$ with trigonometric functions, the first important limit is very often used:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)=1 .$$

The second important limit is used to reveal the uncertainty $\left\{1^{\infty}\right\}$:

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e .$$

Remarks 4.1. Prove independently that $\lim_{z\to 0}(1+z)^{\frac{1}{z}}=e$.

Remark 4.2. When finding specific limits, it is advisable to use the first and second important limits in this form

$$\lim_{x\to x_0}\frac{\sin\alpha(x)}{\alpha(x)}=\lim_{x\to x_0}\frac{\alpha(x)}{\sin\alpha(x)}=1;\ \lim_{x\to x_0}\left(1+\alpha(x)\right)^{1/\alpha(x)}=e,\ \text{ where } \lim_{x\to x_0}\alpha(x)=0.$$

4.3. The concept of equivalent infinitesimal functions. Table of equivalences

Let $\lim_{x\to x_0}\alpha(x)=0$, i.e. $\alpha(x)$ be an infinitesimal function at $x\to x_0$. Then the following equivalences hold in the vicinity of the point $x=x_0$, which are written here with accuracy to the values of the first order of smallness (except for the function $\cos\alpha$):

1) $\sin\alpha\sim\alpha$;
2) $\operatorname{tg}\alpha\sim\alpha$;
3) $\arcsin\alpha\sim\alpha$;
4) $\operatorname{arctg}\alpha\sim\alpha$;
5) $e^{\alpha}-1\sim\alpha$;
6) $a^{\alpha}-1\sim\alpha\ln a$;
7) $\log_a(1+\alpha)\sim\alpha\log_a e$;
8) $\ln(1+\alpha)\sim\alpha$;
9) $(1+\alpha)^{k}-1\sim k\alpha$; $k>0$;

10) $1-\cos\alpha \sim \dfrac{\alpha^2}{2}$.

Remarks 4.3. It is very easy to prove these equivalences using the Lhospital–Bernoulli rule (see below in the Section "Differential calculus of FOV").

Remark 4.4. The above formulas 1)−10) are used when finding equivalences of more complex functions. For example, around a point $x=0$ the function $f(x)=\sin\left[\ln\left(1+\sqrt[3]{\operatorname{tg} x^2}\right)\right]$ is equivalent to:

$$f(x) \sim \sin\left[\ln\left(1+x^{2/3}\right)\right] \sim \sin x^{2/3} \sim x^{2/3}.$$

Remark 4.5. The above equivalences should be used very "***carefully***" from the point of view of mathematical correctness. The choice of the number of members in the representation of the investigated function through equivalent expressions also follows from the condition of each problem, taking into account its context.

4.4. The technique of revealing the main uncertainties

In the simplest cases, finding the limit $\lim\limits_{x\to x_0} f(x)$ is reduced to substituting the limit value of the argument x_0 into the function $f(x)$. But often such a substitution leads to ***uncertain expressions***. Let's consider some of them:

1) ***the ratio*** of two infinite quantities forms the "classical" uncertainty of the form $\{\infty/\infty\}$;

2) ***the difference*** between two infinitely large values is the uncertainty of the form $\{\infty-\infty\}$;

3) ***the product*** of an infinitely small function by an infinitely large one is the uncertainty of the form $\{0\cdot\infty\}$;

4) ***the ratio*** of two ***infinitely small*** quantities ($\alpha_1(x)\to 0$ and $\alpha_2(x)\to 0$ at $x\to x_0$) forms a "***classical***" uncertainty of the form $\{0/0\}$;

5) if $\alpha_1(x)\to 0$ and $\alpha_2(x)\to 0$ at $x\to x_0$, then the expression $\alpha_1(x)^{\alpha_2(x)}$ forms an indeterminacy of the form $\{0^0\}$;

6) if $\alpha(x)\to 0$ and $\beta(x)\to\infty$ at $x\to x_0$, then the expression $\beta(x)^{\alpha(x)}$ is uncertainty of the form $\{\infty^0\}$;

7) if $f(x) \to 1$ and $\beta(x) \to \infty$ at $x \to x_0$, then the expression $f(x)^{\beta(x)}$ is an uncertainty of the form $\{1^\infty\}$.

Definition. *The operation of finding the limit in these cases is called **uncertainty disclosure**.*

Let's consider some individual cases of disclosure of the above-mentioned uncertainties.

1. *Uncertainty of the form* $\{\infty/\infty\}$ *given by the ratio of two polynomials*.

Example 4.1. Find the limit: $\lim\limits_{x\to\infty} \dfrac{x^5 + 3x^2 + 6}{2 - 3x^3 + 10x^2 + 4x^5}$.

***The solution*.** We have "classical" uncertainty of the form $\{\infty/\infty\}$. Let's divide the numerator and denominator by x^5:

$$\lim_{x\to\infty} \frac{x^5 + 3x^2 + 6}{2 - 3x^3 + 10x^2 + 4x^5} = \lim_{x\to\infty} \frac{x^5\left(1 + \dfrac{3}{x^3} + \dfrac{6}{x^5}\right)}{x^5\left(\dfrac{2}{x^5} - \dfrac{3}{x^2} + \dfrac{10}{x^3} + 4\right)} = \lim_{x\to\infty} \frac{1 + \dfrac{3}{x^3} + \dfrac{6}{x^5}}{\dfrac{2}{x^5} - \dfrac{3}{x^2} + \dfrac{10}{x^3} + 4} = \frac{1}{4}.$$

The applied approach is ***general***: to reveal the uncertainty of the form $\{\infty/\infty\}$ given by the ratio of two polynomials, the numerator and denominator must be divided by x^β, where β is equal to the ***highest order*** of the fractional polynomials.

2. *Uncertainty of the form* $\{0/0\}$ *given by the ratio of two polynomials*.

Example 4.2. Find the limit: $\lim\limits_{x\to 3} \dfrac{x^3 - 3x^2 + x - 3}{x^2 - 5x + 6}$.

***The solution*.** Since $\lim\limits_{x\to 3}\left(x^3 - 3x^2 + x - 3\right) = 0$ and $\lim\limits_{x\to 3}\left(x^2 - 5x + 6\right) = 0$, we have uncertainty of the form $\{0/0\}$. To reveal this uncertainty, let's factor the numerator and denominator:

$$\left(x^3 - 3x^2 + x - 3\right) = (x-3)\left(x^2 + 1\right);\ \left(x^2 - 5x + 6\right) = (x-3)(x-2).$$

As a result, we have:

$$\lim_{x\to 3} \frac{x^3 - 3x^2 + x - 3}{x^2 - 5x + 6} = \lim_{x\to 3} \frac{(x-3)\left(x^2 + 1\right)}{(x-3)(x-2)} = \lim_{x\to 3} \frac{x^2 + 1}{x - 2} = 10.$$

This is also a general approach. The factor (in this example, it is $x - 3$), through which the numerator and denominator go to zero, is sometimes called the ***critical factor***.

Therefore, in order to reveal the uncertainty $\{0/0\}$ given by the ratio of two polynomials, it is necessary to select a critical factor in the numerator and in the denominator and reduce the fraction to it. If at the same time the factorization turns out to be difficult, then it is necessary to divide the numerator and denominator into a critical factor by a "corner". At the same time, the hint, which is the critical factor, is contained in the task itself under the icon $\lim\limits_{x\to a}$ ($x \to a$ $\Rightarrow (x-a)\to 0$).

3. ***Uncertainty*** $\{0/0\}$ ***given by irrational expressions***.

Example 4.3. Find the limit: $\lim\limits_{x\to 3}\dfrac{\sqrt{x^2+7}-4}{x-3}$.

The solution. Here we have uncertainty of the classical form $\{0/0\}$, and $(x-3)$ – is the critical factor. Let's get rid of the irrationality in the numerator by multiplying both the numerator and the denominator by the conjugated expression. As a result, we get

$$\lim_{x\to 3}\frac{\sqrt{x^2+7}-4}{x-3}=\lim_{x\to 3}\frac{\left(\sqrt{x^2+7}-4\right)\left(\sqrt{x^2+7}+4\right)}{(x-3)\left(\sqrt{x^2+7}+4\right)}=\lim_{x\to 3}\frac{x^2-9}{(x-3)\left(\sqrt{x^2+7}+4\right)}=$$

$$=\lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)\left(\sqrt{x^2+7}+4\right)}=\lim_{x\to 3}\frac{x+3}{\sqrt{x^2+7}+4}=\frac{3}{4}.$$

4. ***Uncertainty*** $\{\infty-\infty\}$ ***given by irrational expressions.***

Example 4.4. Find the limit: $\lim\limits_{x\to+\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-5x+7}\right)$.

The solution.

$$\lim_{x\to+\infty}\left(\sqrt{x^2+3x}-\sqrt{x^2-5x+7}\right)=$$

$$=\lim_{x\to+\infty}\frac{\left(\sqrt{x^2+3x}-\sqrt{x^2-5x+7}\right)\left(\sqrt{x^2+3x}+\sqrt{x^2-5x+7}\right)}{\sqrt{x^2+3x}+\sqrt{x^2-5x+7}}=\lim_{x\to+\infty}\frac{x^2+3x-x^2+5x-7}{\sqrt{x^2+3x}+\sqrt{x^2-5x+7}}=$$

$$=\lim_{x\to+\infty}\frac{8x-7}{\sqrt{x^2+3x}+\sqrt{x^2-5x+7}}=\lim_{x\to+\infty}\frac{8-\frac{7}{x}}{\sqrt{1+\frac{3}{x}}+\sqrt{1-\frac{5}{x}+\frac{7}{x^2}}}=4\cdot$$

5. ***Uncertainties of the type*** $\{0/0\}$ ***given by expressions containing trigonometric functions are often revealed by means of the first significant limit.***

Example 4.5. Find the limit: $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{x^3}$.

The solution.

$$\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}=\left\{\frac{0}{0}\right\}=\lim_{x\to 0}\frac{2\sin x\cdot(1-\cos x)}{x^3}=$$

$$=2\lim_{x\to 0}\frac{\sin x\cdot 2\sin^2\dfrac{x}{2}}{x^3}=4\lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\left(\dfrac{x}{2}\right)^2}{x^2}=4\cdot 1\cdot\frac{1}{4}=1.$$

6. ***When revealing the uncertainty of the species*** $\{1^\infty\}$, ***the second important limit is used.***

Example 4.6. Find the limit: $\lim\limits_{x\to 0}(\cos x)^{(\sin x)^{-2}}$.

The solution.

$$\lim_{x\to 0}(\cos x)^{(\sin x)^{-2}}=\{1^\infty\}=\lim_{x\to 0}(1+\cos x-1)^{\frac{1}{\cos x-1}\cdot\frac{\cos x-1}{(\sin x)^2}}=e^{\lim\limits_{x\to 0}\left(\frac{\cos x-1}{(\sin x)^2}\right)}=e^{-\lim\limits_{x\to 0}\left(\frac{x^2}{2x^2}\right)}=\frac{1}{\sqrt{e}}.$$

Let's consider a number of examples of different types on the topic of calculating limits.

Example 4.7. Find the limit $\lim\limits_{x\to 0}\left(\dfrac{(1+x)^n-1}{x}\right)$.

The solution. Let's establish the type of uncertainty that arises when finding a given limit. By substituting the sign of the limit instead of x zero, we obtain an uncertainty of the type $\left\{\dfrac{0}{0}\right\}$. Next, we will open the brackets for Newton's binomial and simplify the resulting expression:

$$\lim_{x\to 0}\left(\frac{(1+x)^n-1}{x}\right)=\left\{\frac{0}{0}\right\}=\lim_{x\to 0}\left(\frac{1+nx+C_n^2x^2+\cdots+x^n-1}{x}\right)=\lim_{x\to 0}\left(n+C_n^2x+\cdots+x^{n-1}\right)=n$$

Independent work of students. Solve this example again after studying the ***Lhospital–Bernoulli*** method and compare the two approaches.

Example 4.8. Find the limit $\lim\limits_{x\to 0}\left(\dfrac{\sqrt[m]{1+x}-1-\dfrac{x}{m}}{x^2}\right)$.

The solution. To find the limit, we will replace the variables:

$$\lim_{x\to 0}\left(\frac{\sqrt[m]{1+x}-1-\dfrac{x}{m}}{x^2}\right)=\left\{\frac{0}{0}\right\}=\left|\begin{matrix} y=\sqrt[m]{1+x}-1; x+1=(1+y)^m \\ y\to 0;\ x=(1+y)^m-1\end{matrix}\right|=$$

$$=\lim_{y\to 0}\left(\frac{y-\dfrac{(1+y)^m-1}{m}}{[(1+y)^m-1]^2}\right)=\lim_{y\to 0}\left(\frac{y-\dfrac{1}{m}\left[1+my+\dfrac{m(m-1)}{2}y^2+\cdots+y^m-1\right]}{\left[1+my+\dfrac{m(m-1)}{2}y^2+\cdots+y^m-1\right]^2}\right)=$$

$$=\lim_{y\to 0}\left(\frac{-\dfrac{(m-1)}{2}y^2}{m^2y^2}\right)=\frac{1-m}{2m^2}.$$

Independent work of students. Solve this example again after studying the ***Lhospital–Bernoulli*** method and compare the two approaches.

Example 4.9. Find the limit $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{1+x^2+5x^3}-1}{1-\cos 2x+6x^2}\right)$.

The solution. We multiply the numerator and denominator by the conjugate expression $(\sqrt{1+x^2+5x^3}+1)$ with respect to the numerator and replace the expression $(1-\cos 2x)$ with its equivalent representation:

$$\lim_{x\to 0}\left(\frac{\sqrt{1+x^2+5x^3}-1}{1-\cos 2x+6x^2}\right)=\left\{\frac{0}{0}\right\}=\lim_{x\to 0}\left(\frac{1+x^2+5x^3-1}{(2\sin^2 x+6x^2)\left(\sqrt{1+x^2+5x^3}+1\right)}\right)=$$

$$=\left|\begin{matrix}\sin x\sim x\\ x\to 0\end{matrix}\right|=\lim_{x\to 0}\left(\frac{x^2+5x^3}{(2x^2+6x^2)\left(\sqrt{1+x^2+5x^3}+1\right)}\right)=\lim_{x\to 0}\left(\frac{x^2}{8x^2\cdot 2}\right)=\frac{1}{16}.$$

Example 4.10. Find the limit $\lim\limits_{x\to 0}\left(\dfrac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2+\sin^3 x}\right)$.

The solution.

$$\lim_{x\to 0}\left(\frac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2+\sin^3 x}\right)=\left\{\frac{0}{0}\right\}=\lim_{x\to 0}\left(\frac{(\sqrt{1+x}-1)+(\sqrt{1-x}-1)}{x^2+\sin^3 x}\right)=$$

$$=\lim_{x\to 0}\left(\frac{\dfrac{x}{\sqrt{1+x}+1}-\dfrac{x}{\sqrt{1-x}+1}}{x^2+\sin^3 x}\right)=\lim_{x\to 0}\left(\frac{\dfrac{x\left(\sqrt{1-x}-\sqrt{1+x}\right)\left(\sqrt{1-x}+\sqrt{1+x}\right)}{(\sqrt{1+x}+1)(\sqrt{1-x}+1)\left(\sqrt{1-x}+\sqrt{1+x}\right)}}{x^2+\sin^3 x}\right)=$$

$$=\lim_{x\to 0}\left(\frac{\dfrac{-2x^2}{8}}{x^2}\right)=-\frac{1}{4}.$$

Example 4.11. Find the limit

$$\lim_{x\to 0}\left(\frac{\sqrt[4]{1+8x-4\sin^2(x)+6\,\mathrm{tg}(x^3)+12x^4}-\sqrt[5]{1+10x-5\,\mathrm{tg}(x^2)+5\sin^3(x)+16x^4}}{4x^3}\right).$$

The solution. We have uncertainty $\left\{\frac{0}{0}\right\}$. To find the limit, we will use the table of equivalences (No 1, 2, 9):

$$\lim_{x\to 0}\left(\frac{\sqrt[4]{1+8x-4\sin^2 x+6\,\mathrm{tg}(x^3)+12x^4}-\sqrt[5]{1+10x-5\,\mathrm{tg}(x^2)+5\sin^3 x+20x^4}}{4x^3}\right)=$$

$$=\left|\begin{matrix}(1+\alpha)^k\sim 1+\alpha k,\\ if\ \ \alpha\to 0\end{matrix}\right|=\lim_{x\to 0}\left(\frac{1+2x-\sin^2 x+\frac{3}{2}\mathrm{tg}(x^3)+3x^4-[1+2x-\mathrm{tg}(x^2)+\sin^3 x+4x^4]}{4x^3}\right)=$$

$$=\lim_{x\to 0}\left(\frac{-\sin^2(x)+\dfrac{3}{2}\mathrm{tg}(x^3)+3x^4+\mathrm{tg}(x^2)-\sin^3(x)-4x^4}{4x^3}\right)=$$

$$=\lim_{x\to 0}\left(\frac{-x^2+\dfrac{3}{2}x^3+x^2-x^3-x^4}{4x^3}\right)=\lim_{x\to 0}\left(\frac{\dfrac{1}{2}x^3-x^4}{4x^3}\right)=\frac{1}{8}.$$

Example 4.12. Find the limit $\lim\limits_{x\to 0}\left(\dfrac{a^{x+1}+b^{x+1}+c^{x+1}+d^{x+1}}{a+b+c+d}\right)^{\frac{1}{x}}$, where a,b,c,d – positive real numbers are fixed.

The solution. We have uncertainty $\{1^{\infty}\}$. Let's make the following transformations:

$$\lim_{x\to 0}\left(\frac{a^{x+1}+b^{x+1}+c^{x+1}+d^{x+1}}{a+b+c+d}\right)^{\frac{1}{x}}=\{1^{\infty}\}=$$

$$=\lim_{x\to 0}\left(1+\frac{(a^{x+1}-a)+(b^{x+1}-b)+(c^{x+1}-c)+(d^{x+1}-d)}{a+b+c+d}\right)^{\frac{1}{x}}=\left|\begin{matrix}\alpha^{x}-1\sim x\ln\alpha,\\ if\;\; x\to 0\end{matrix}\right|=$$

$$=\lim_{x\to 0}\left(1+\frac{a\ln a+b\ln b+c\ln c+d\ln d}{a+b+c+d}\cdot x\right)^{\frac{1}{x}}=\lim_{x\to 0}\left(1+\frac{\ln(a^{a}b^{b}c^{c}d^{d})}{a+b+c+d}\cdot x\right)^{\frac{1}{x}}=$$

$$=e^{\frac{\ln(a^{a}b^{b}c^{c}d^{d})}{a+b+c+d}}=(a^{a}b^{b}c^{c}d^{d})^{\frac{1}{a+b+c+d}}.$$

Example 4.13. Find the limit $\lim\limits_{x\to 81}\left(\dfrac{x^{n}-81^{n}-n\cdot 81^{n-1}(x-81)}{(x-81)^{2}}\right)$, where n – is a natural number.

The solution. We have classical uncertainty $\left\{\dfrac{0}{0}\right\}$. Let's replace the variable as follows: $x=z+81$, where under the sign of the limit should be written: $z\to 0$. Then the given limit will look like this:

$$\lim_{z\to 0}\left(\frac{(z+81)^{n}-81^{n}-n\cdot 81^{n-1}z}{z^{2}}\right).$$

We open the brackets in the numerator according to Newton's binomial and after identical transformations we get:

$$\lim_{z\to 0}\left(\frac{(z+81)^{n}-81^{n}-n\cdot 81^{n-1}z}{z^{2}}\right)=$$

$$=\lim_{z\to 0}\left(\frac{(z^{n}+81\cdot n\cdot z^{n-1}+81^{2}\cdot C_{n}^{2}\cdot z^{n-2}+\cdots+81^{n-2}\cdot C_{n}^{n-2}\cdot z^{2}+n\cdot 81^{n-1}z+81^{n})-81^{n}-n\cdot 81^{n-1}z}{z^{2}}\right)=$$

$$=\lim_{z\to 0}\left(\frac{z^{n}+81\cdot n\cdot z^{n-1}+81^{2}\cdot C_{n}^{2}\cdot z^{n-2}+\cdots+81^{n-2}\cdot C_{n}^{n-2}\cdot z^{2}}{z^{2}}\right)=$$

$$=\lim_{z\to 0}\left(\frac{z^{n-2}+81\cdot n\cdot z^{n-3}+\cdots+81^{n-3}\cdot C_{n}^{n-3}\cdot z+81^{n-2}\cdot C_{n}^{n-2}}{1}\right)=81^{n-2}\cdot C_{n}^{n-2}=\frac{n(n-1)}{2}\cdot 81^{n-2}.$$

Independent work of students. Solve this example again after studying the ***Lhospital–Bernoulli*** method and compare the two approaches.

Example 4.14. Find the limit $\lim\limits_{x\to 0}\left(\dfrac{81^{x+p}+81^{p-x}-2\cdot 81^{p}}{x^2}\right)$, where $p-$ is a fixed real number.

The solution. Let's put the common factor in parentheses in the numerator and make identical transformations using equivalences up to the members $o(x^2)$:

$$\lim_{x\to 0}\left(\frac{81^{x+p}+81^{p-x}-2\cdot 81^{p}}{x^2}\right)=81^{p}\lim_{x\to 0}\left(\frac{(81^{x}-1)+(81^{-x}-1)}{x^2}\right)=$$

$$=\left|a^{x}=e^{x\ln a}\sim 1+x\cdot\ln a+\frac{(x\cdot\ln a)^2}{2!}+\cdots\right|=$$

$$=81^{p}\lim_{x\to 0}\left(\frac{\left(x\cdot\ln 81+\dfrac{(x\cdot\ln 81)^2}{2!}+\cdots\right)+\left(-x\cdot\ln 81+\dfrac{(x\cdot\ln 81)^2}{2!}-\cdots\right)}{x^2}\right)=$$

$$=81^{p}\lim_{x\to 0}\left(\frac{\left(2\cdot\dfrac{(x\cdot\ln 81)^2}{2!}+\cdots\right)}{x^2}\right)=81^{p}\cdot\ln^2 81=81^{p}\cdot 16\cdot\ln^2 3\,.$$

Here, the terms that have a higher order of smallness than x^2, are marked with dots.

Independent work of students. Solve this example again after studying the ***Lhospital–Bernoulli*** method and compare the two approaches.

Example 4.15. Find the limit $\lim\limits_{x\to 0}\left(\dfrac{(a+x)^{x}-a^{x}}{x^2}\right)$, where $a>0$ – is a fixed real number.

The solution. Let's make some transformations, after which we will find the limit:

$$\lim_{x\to 0}\left(\frac{a^{x}\left[\left(1+\dfrac{x}{a}\right)^{x}-1\right]}{x^2}\right)=\left\{\frac{0}{0}\right\}=\lim_{x\to 0}\left(\frac{\left[e^{\frac{x^2}{a}}-1\right]}{x^2}\right)=\left|e^{\frac{x^2}{a}}-1\sim\frac{x^2}{a}\right|=\lim_{x\to 0}\left(\frac{\dfrac{x^2}{a}}{x^2}\right)=\frac{1}{a}\,.$$

Example 4.16. Find the limit $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$.

The solution. Let's use the equivalence for the function with accuracy up to the second term:

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} = \left\{1^{\infty}\right\} = \left|\begin{matrix}\sin x \sim x - \frac{x^3}{6} + \cdots, \\ if \;\; x\to 0\end{matrix}\right| = \lim_{x\to 0}\left(\frac{x - \frac{x^3}{6}}{x}\right)^{\frac{1}{x^2}} = \lim_{x\to 0}\left(1 - \frac{x^2}{6}\right)^{\frac{1}{x^2}} = e^{-\frac{1}{6}}.$$

Example 4.17. Find the limit $\lim\limits_{x\to +\infty}\left((x+a)^{1+\frac{1}{x}} - x^{1+\frac{1}{x+a}}\right)$, where a – is an arbitrary fixed real number.

The solution. Let's make some transformations, after which we will find the limit:

$$\lim_{x\to +\infty}\left((x+a)^{1+\frac{1}{x}} - x^{1+\frac{1}{x+a}}\right) = \left\{\infty - \infty\right\} = \lim_{x\to +\infty} x^{1+\frac{1}{x}}\left(\left(1+\frac{a}{x}\right)^{1+\frac{1}{x}} - x^{\frac{1}{x+a}-\frac{1}{x}}\right) =$$

$$= \lim_{x\to +\infty} x\cdot\left(\left(1+\frac{a}{x}\right)^{1+\frac{1}{x}} - x^{\frac{1}{x+a}-\frac{1}{x}}\right) = \lim_{x\to +\infty} x\cdot\left(e^{\frac{a}{x}+\frac{a}{x^2}} - x^{-\frac{a}{(x+a)x}}\right) =$$

$$= \lim_{x\to +\infty} x\cdot\left(e^{\frac{a}{x}+\frac{a}{x^2}} - x^{-\frac{a}{(x+a)x}}\right) = \lim_{x\to +\infty} x\cdot\left(1+\frac{a}{x}+\frac{a}{x^2} - e^{-\frac{a}{(x+a)x}\cdot\ln x}\right) =$$

$$= \lim_{x\to +\infty} x\cdot\left(1+\frac{a}{x}+\frac{a}{x^2} - \left(1 - \frac{a\cdot\ln x}{(x+a)x} + \frac{1}{2}\cdot\left(\frac{a\cdot\ln x}{(x+a)x}\right)^2\right)\right) =$$

$$= \lim_{x\to +\infty}\left(a + \frac{a}{x} + \frac{a\cdot\ln x}{x+a} - \frac{1}{2x}\cdot\left(\frac{a\cdot\ln x}{x+a}\right)^2\right) = a$$

Example 4.18. Find the limit $\lim\limits_{x\to a}\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}\right)^{\operatorname{ctg}(x-a)}$, where $a \neq \frac{\pi k}{2}$ – is an arbitrary fixed real number.

The solution. Let's make some transformations, after which we will find the limit:

$$\lim_{x\to a}\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}\right)^{\operatorname{ctg}(x-a)}=\left\{1^{\infty}\right\}=\lim_{x\to a}\left(1+\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}-1\right)\right)^{\frac{1}{\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}-1\right)}\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}-1\right)\operatorname{ctg}(x-a)}=$$

$$=\lim_{x\to a}\left(1+\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}-1\right)\right)^{\frac{1}{\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}-1\right)}\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}-1\right)\operatorname{ctg}(x-a)}=e^{\lim\limits_{x\to a}\left(\left(\frac{\operatorname{tg} x}{\operatorname{tg} a}-1\right)\operatorname{ctg}(x-a)\right)}=$$

$$=\left|\begin{array}{l}\lim\limits_{x\to a}\left(\left(\dfrac{\operatorname{tg} x-\operatorname{tg} a}{\operatorname{tg} a}\right)\operatorname{ctg}(x-a)\right)=\\ =\lim\limits_{x\to a}\left(\dfrac{\sin(x-a)}{\sin a\cdot\cos x}\cdot\dfrac{\cos(x-a)}{\sin(x-a)}\right)=\dfrac{1}{\sin a\cdot\cos a}\end{array}\right|=e^{\frac{2}{\sin 2a}}.$$

Tasks for classroom and independent work

Find the limit:

4.1. $\lim\limits_{x\to\infty}\dfrac{1+x^2-x^3}{2x^4+3}$;

4.2. $\lim\limits_{x\to\infty}\dfrac{4x^5+3x^2-1}{x^3-2}$;

4.3. $\lim\limits_{x\to\infty}\dfrac{x^3-4x+1}{2x^3-x^2+8}$.

4.4. $\lim\limits_{x\to\infty}\dfrac{3x^4+2}{\sqrt{x^8-2x+1}}$.

4.5. $\lim\limits_{x\to\infty}\dfrac{\sqrt[3]{x^3+1}+\sqrt{9x^4+5}}{\sqrt[4]{5x^4+3}+\sqrt[5]{x^{10}+3}}$.

4.6. $\lim\limits_{x\to\infty}\dfrac{(x-1)(x-2)(x-3)(x-4)(x-5)}{(5x-1)^5}$.

4.7. $\lim\limits_{x\to\infty}\dfrac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}}$.

4.8. $\lim\limits_{x\to 1}\dfrac{x^2+x-2}{x^3-x^2-x+1}$

4.9. $\lim\limits_{x\to 0}\dfrac{(1+x)^5-(1+5x)}{x^2+x^5}$.

4.10. $\lim\limits_{x\to 0}\dfrac{(1+mx)^n-(1+nx)^m}{x^2},\ m,n\in\mathbb{N}$.

4.11. $\lim\limits_{x\to 1}\dfrac{x^3-3x+2}{x^4-4x+3}$.

4.12. $\lim\limits_{x\to 2}\dfrac{x^3-2x^2-4x+8}{x^4-8x^2+16}$.

4.13. $\lim\limits_{x\to -1}\dfrac{x^3-2x-1}{x^5-2x-1}$.

4.14. $\lim\limits_{x\to 1}\dfrac{x^{100}-2x+1}{x^{50}-2x+1}$.

4.15. $\lim\limits_{x\to 1}\dfrac{x^{n+1}-(n+1)x+n}{(x-1)^2},\ n\in\mathbb{N}$.

4.16. $\lim\limits_{x\to 4}\dfrac{\sqrt{2x+1}-3}{\sqrt{x}-2}$.

4.17. $\lim\limits_{x\to 1}\dfrac{2x^2-x-1}{\sqrt{5x+1}-\sqrt{4x+2}}$.

4.18. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+5x}-\sqrt{1-5x}}{x^2+7x}$.

4.19. $\lim\limits_{x\to 0}\dfrac{4-\sqrt{x^2+16}}{x^2}$.

4.20. $\lim\limits_{x\to -8}\dfrac{\sqrt{1-x}-3}{\sqrt[3]{x}+2}$.

4.21. $\lim\limits_{x\to 3}\dfrac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}$.

4.22. $\lim\limits_{x\to -2}\dfrac{\sqrt[3]{x-6}+2}{x^3+8}$.

4.23. $\lim\limits_{x\to 8}\dfrac{\sqrt{2x+9}-5}{\sqrt[3]{x}-2}$.

4.24. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{8+3x-x^2}-2}{x^2+x}$.

4.25. $\lim\limits_{x\to 7}\dfrac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2}$.

4.26. $\lim\limits_{x\to 2}\left(\dfrac{1}{x-2}-\dfrac{12}{x^3-8}\right)$.

4.27. $\lim\limits_{x\to 1}\left(\dfrac{3}{1-\sqrt{x}}-\dfrac{2}{1-\sqrt[3]{x}}\right)$.

4.28. $\lim\limits_{x\to 1}\dfrac{(1-\sqrt{x})(1-\sqrt[3]{x})\cdots(1-\sqrt[n]{x})}{(1-x)^{n-1}}$.

4.29. $\lim\limits_{x\to -\infty}\left(\sqrt{x^2+8x+3}-\sqrt{x^2+4x+3}\right)$.

4.30. $\lim\limits_{x\to \pm\infty}\left(2x-\sqrt{4x^2-7x+4}\right)$.

4.31. $\lim\limits_{x\to \pm\infty}\left(\sqrt{4x^2+3x+1}-\sqrt{4x^2-5x+3}\right)$.

4.32. $\lim\limits_{x\to \pm\infty}\left(\sqrt{(x+a)(x+b)}-x\right)$.

4.33. $\lim\limits_{x\to +\infty}\dfrac{(x-\sqrt{x^2-1})^n+(x+\sqrt{x^2-1})^n}{x^n}$, $n\in\mathbb{N}$.

4.34. $\lim\limits_{x\to 0}\dfrac{(\sqrt{1+x^2}+x)^n-(\sqrt{1+x^2}-x)^n}{x}$, $n\in\mathbb{N}$.

4.35. $\lim\limits_{x\to 0}\dfrac{\cos 7x-\cos 3x}{1-\cos 2x}$.

4.36. $\lim\limits_{x\to 0}\dfrac{\cos x-\cos^3 x}{x\arcsin 3x}$.

4.37. $\lim\limits_{x\to 0}\dfrac{\operatorname{tg} x-\sin x}{\sin^3 x}$.

4.38. $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 3x}{\sin x}$.

4.39. $\lim\limits_{x\to \frac{\pi}{4}}\operatorname{tg} 2x\cdot\operatorname{tg}\left(\dfrac{\pi}{4}-x\right)$.

4.40. $\lim\limits_{x\to 1}(1-x)\operatorname{tg}\left(\dfrac{\pi x}{2}\right)$.

4.41. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{4x+1}}{\cos\dfrac{\pi(x+1)}{2}}$.

4.42. $\lim\limits_{x\to \frac{\pi}{4}}\dfrac{(\pi-4x)^2}{1-\sin 2x}$.

4.43. $\lim\limits_{x\to \frac{\pi}{3}}\dfrac{\operatorname{tg}^3 x-3\operatorname{tg} x}{\cos\left(x+\dfrac{\pi}{6}\right)}$.

4.44. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-\sqrt{\cos 2x}}{\operatorname{tg}^2\dfrac{x}{2}}$.

4.45. $\lim\limits_{x\to\infty} x^2\left(1-\cos\frac{1}{x}\right)$.

4.46. $\lim\limits_{x\to\infty}\left(\cos\sqrt{x+1}-\cos\sqrt{x}\right)$.

4.47. $\lim\limits_{x\to\infty}\left(\frac{x-1}{x+2}\right)^{\frac{x+1}{2}}$.

4.48. $\lim\limits_{x\to\infty}\left(\frac{x^2-2x+8}{x^2-2x-7}\right)^{x^2}$.

4.49. $\lim\limits_{x\to 0}\left(1+\mathrm{tg}^2\sqrt{x}\right)^{\frac{1}{2x}}$.

4.50. $\lim\limits_{x\to\frac{\pi}{4}}(tgx)^{tg\,2x}$.

4.51. $\lim\limits_{x\to 0}\frac{x}{\ln(x+2)-\ln 2}$.

4.52. $\lim\limits_{x\to 3}\frac{e^x-e^3}{x-3}$.

4.53. $\lim\limits_{x\to 0}\frac{e^{\sin 2x}-e^{\sin x}}{x}$.

4.54. $\lim\limits_{x\to+\infty}\frac{\ln(1+\sqrt{x}+\sqrt[3]{x})}{\ln(1+\sqrt[3]{x}+\sqrt[4]{x})}$.

4.55. $\lim\limits_{x\to 0}\frac{\ln\mathrm{tg}\left(\frac{\pi}{4}+ax\right)}{\sin bx}$.

4.56. $\lim\limits_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx}$.

4.57. $\lim\limits_{x\to 0}\left(x+e^x\right)^{\frac{1}{x}}$.

4.58. $\lim\limits_{x\to 1}\frac{\sin(\pi x^{\alpha})}{\sin(\pi x^{\beta})}$.

4.59. $\lim\limits_{x\to+\infty}\ln(1+2^x)\ln\left(1+\frac{3}{x}\right)$.

4.60. $\lim\limits_{x\to 0}\frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$.

4.61. $\lim\limits_{x\to 0}\left(2e^{\frac{x}{x+1}}-1\right)^{\frac{x^2+1}{x}}$.

4.62. $\lim\limits_{h\to 0}\frac{\mathrm{arctg}(x+h)-\mathrm{arctg}\,x}{h}$.

4.63. Find constants *a* and *b*, if $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x+1}-ax-b\right)=0$.

4.64. Find $h=\lim\limits_{x\to+\infty}f(x)-\lim\limits_{x\to-\infty}f(x)$, if $f(x)=\ln\frac{x+\sqrt{x^2+a^2}}{x+\sqrt{x^2+b^2}}$.

4.65. $\lim\limits_{x\to 0}\left(\frac{e^x\sin x-x(1+x)}{x^3}\right)$.

4.66. $\lim\limits_{x\to 0}\left(\frac{a^x-a^{\sin x}}{\sin\left(x^3\right)}\right), a>0$.

4.67. $\lim\limits_{x\to 0}\left(\sqrt[x]{\frac{2^{4x}\cdot 5^{2x}+3^{3x}\cdot 6^x}{2}}\right)$.

4.68. $\lim\limits_{x\to+\infty}\left(\sqrt[200]{(x+1)\cdot(x+2)\cdot\dots\cdot(x+200)}-x\right)$.

4.69. $\lim\limits_{x\to+\infty}\left(\sqrt[100]{(x+3\cdot 1)\cdot(x+3\cdot 2)\cdot\dots\cdot(x+3\cdot 100)}-x\right)$.

4.70. $\lim\limits_{x\to 0}\dfrac{\sin\left(3^x-2^x\right)^5}{\operatorname{arctg}\left(3^{x^5}-2^{x^5}\right)}$.

4.71. $\lim\limits_{x\to 0}\dfrac{\left(\sqrt{1+2x^2+4x^3}+x\right)^n-\left(\sqrt{1+2x^2-2x^3}+x\right)^n}{3x^3}$.

4.72. $\lim\limits_{x\to 0}\left[x^{-2}\cdot\left(\left(1+m\cdot x+\dfrac{x^2}{n}\right)^n-\left(1+nx+\dfrac{x^2}{m}\right)^m\right)\right]$.

4.73. $\lim\limits_{x\to 0}\left(\dfrac{\arcsin x}{x}\right)^{\frac{1}{x^2}}$.

4.74. $\lim\limits_{x\to 0}\left(\dfrac{1+\sin x\cdot\cos(\alpha x)}{1+\sin x\cdot\cos(\beta x)}\right)^{\operatorname{ctg}(x^3)}$.

4.75. $\lim\limits_{x\to 0}\left(15-\dfrac{1-\cos(x^2)\cos(2x^2)\cos(3x^2)}{1-\cos(x^2)}\right)^{\frac{2\operatorname{ctg}(x^4)}{9}}$.

Answers and instructions

4.1. 0. **4.2.** ∞. **4.3.** $1/2$.

4.4. 3. **4.5.** 3. **4.6.** 5^{-5}.

4.7. $(3/2)^{30}$. **4.8.** ∞. **4.9.** 10.

4.10. $nm(n-m)/2$. **4.11.** $1/2$. **4.12.** $1/4$.

4.13. $1/3$. **4.14.** $49/24$. **4.15.** $n(n+1)/2$.

4.16. $4/3$. **4.17.** $6\sqrt{6}$. **4.18.** $5/7$.

4.19. $-1/8$. **4.20.** -2. **4.21.** $-1/16$.

4.22. $1/144$. **4.23.** $12/5$. **4.24.** $1/4$.

4.25. $112/27$. **4.26.** $1/2$. **4.27.** $1/2$.

4.28. $1/n!$. **4.29.** -2. **4.30.** $7/4$.

4.31. ± 2. **4.32.** $(a+b)/2$. **4.33.** 2^n.

4.34. $2n$. **4.35.** -10. **4.36.** $1/3$.

4.37. $1/2$. **4.38.** 2. **4.39.** $1/2$.

4.40. $2/\pi$. **4.41.** $4/\pi$. **4.42.** 8.

4.43. -24. **4.44.** 6. **4.45.** $1/2$.

4.46. 0. **4.47.** $e^{-1,5}$. **4.48.** e^{15}.

4.49. $\sqrt{e}$. **4.50.** $1/e$. **4.51.** 2.

4.52. e^3. **4.53.** 1. **4.54.** $3/2$.

4.55. $2a/b$. **4.56.** $(a/b)^2$. **4.57.** e^2.

4.58. α/β. **4.59.** $\ln 8$. **4.60.** $1/2$.

4.61. e^2. **4.62.** $1/(1+x^2)$. **4.63.** $a=1,\ b=-1$.

4.64. $2\ln(b/a)$. **4.65.** $\frac{1}{3}$. **4.66.** $\frac{1}{6}\ln a$.

4.67. $180\sqrt{2}$. **4.68.** $\frac{201}{2}$. **4.69.** $\frac{303}{2}$.

4.70. $\left[\ln\left(\frac{2}{3}\right)\right]^4$. **4.71.** n. **4.72.** $\frac{nm(n-m)}{2}$.

4.73. $e^{\frac{1}{6}}$. **4.74.** $e^{\frac{\beta^2-\alpha^2}{2}}$. **4.75.** e^7.

Topic III. The concept of continuity of functions. Study of functions for continuity. Classification of breakpoints. Uniform continuity of functions

PRACTICAL LESSON 5
Continuity of a function. Classification of breakpoints. Study of functions for continuity

5.1. Continuity of function at a point. Three definitions of continuity of a function

Let us recall some theoretical definitions related to the concept of continuity of a function. Let the function $f(x)$ be defined at a point x_0 and some neighborhood of it.

The first definition of the continuity. A function $f(x)$ is called continuous at a point x_0 if the limit of the function and its value at this point are equal, i.e.:

$$\lim_{x \to x_0} f(x) = f(x_0). \tag{5.1}$$

If we compare this definition with the definition of the boundary of a function $\lim_{x \to x_0} f(x) = A$, then when defining the boundary of a function, the number x_0 could not belong to the domain of the function (for example, as in the function $y = \frac{\sin x}{x}$ at the point $x_0 = 0$), and in the case that the number x_0 belonged to the domain of the function, then the value of the function $f(x_0)$ could and not coincide with the border A.

Therefore, a function $f(x)$ will be continuous at a point x_0 if and only if the following conditions hold:

a) the function is defined at a point x_0 and some neighborhood of this point;

b) there is a border $\lim_{x \to x_0} f(x)$;

c) the boundary of the function $f(x)$ at a point x_0 and the value of the function at this point coincide, that is, equality (5.1) is fulfilled.

Formula (5.1) can be written in the form

$$\lim_{x \to x_0} f(x) = f\left(\lim_{x \to x_0} x\right) = f(x_0). \tag{5.2}$$

We will give the second definition of the continuity of a function, based on the concept of increments of an argument and a function. Let the numbers x_0 and x belong to the definition domain of the function $y = f(x)$.

Definition. *The difference* $x - x_0$ *is called the increment of the argument at the point* x_0 *and is denoted by* Δx *:* $\Delta x = x - x_0$. *The difference of the corresponding values of the function* $f(x) - f(x_0)$ *is called the increment of the function at the point* x_0 *and is denoted by* Δy *:*

$$\Delta y = \Delta f = f(x) - f(x_0) = f(x_0 + \Delta x) - f(x_0).$$

Obviously, the increment Δx can be a positive or negative number. Let's write equality (5.1) in the new notation, for which we will transfer the value $f(x_0)$ in it to the left part and enter it under the limit sign. Since the conditions $x \to x_0$ and $x - x_0 \to 0$ are the same, equality (5.1) takes the form

$$\lim_{x - x_0 \to 0} \left(f(x) - f(x_0) \right) = 0 \quad \text{or} \quad \lim_{\Delta x \to 0} \Delta y = 0. \qquad (5.3)$$

Equality (5.3) makes it possible to formulate the following definition of the continuity of a function.

Second definition of the continuity. *A function* $y = f(x)$ *defined at a point* x_0 *and some of its neighborhood is called continuous at a point* x_0 *if its increment at this point is an infinitesimally small function at* $\Delta x \to 0$.

The concept of one-sided continuity is often encountered.

Definition of continuity on the left. *A function* $y = f(x)$ *is called continuous at a point* x_0 *on the left if it is defined on the half-interval* $(x_0 - \delta; x_0]$, *where* $\delta > 0$, *and is a valid equality*

$$\lim_{x \to x_0 - 0} f(x) = f(x_0 - 0).$$

Definition of continuity on the right. *A function* $y = f(x)$ *is said to be continuous at a point* x_0 *on the right if the function is defined on the half-interval* $[x_0; x_0 + \delta)$, *where* $\delta > 0$, *and is a valid equality*

$$\lim_{x \to x_0 + 0} f(x) = f(x_0 + 0).$$

Using concepts and formulas (5.2), it is possible to formulate ***the third definition of continuity***.

A function $y = f(x)$ will be continuous at a point x_0 if and only if it is defined in some neighborhood of a point x_0 and the following holds equality:

$$\lim_{x \to x_0 - 0} f(x) = \lim_{x \to x_0 + 0} f(x) = f(x_0). \qquad (5.4)$$

If at least one of these conditions is not fulfilled, then the function is called discontinuous at the point x_0, and the point itself x_0 is called the point of discontinuity of the function.

Definition. *A function* $y = f(x)$ *is called continuous on an interval* (a,b) *if it is continuous at every point of this interval.*

A function $y = f(x)$ is continuous on the segment $[a,b]$, if it is continuous at every point of the interval (a,b) and at point $x = a$ is continuous on the right (that is, $\lim_{x \to a+0} f(x) = f(a)$) and at point $x = b$ is continuous on the left (that is, $\lim_{x \to b-0} f(x) = f(b)$).

Let us give a number of properties that are satisfied by functions that are continuous on a line segment. Each of these properties has an important independent meaning:

1) if the function $y = f(x)$ is continuous on the segment $[a,b]$, then it acquires the largest and smallest values on it;

2) if the function $y = f(x)$ is continuous on the segment $[a,b]$ and acquires values of different signs at its ends, then in the interval (a,b) there is at least one such point c at which the function is zero: $f(c) = 0$;

3) if the function $y = f(x)$ is continuous on the segment $[a,b]$ and *m* is its smallest value, and *M* is the largest, then for any number μ contained between *m* and *M*, there will be such an argument value $c \in [a,b]$ that $f(c) = \mu$;

4) if two functions $f(x)$ and $g(x)$ are defined in the domain *X* and are continuous at the point $x_0 \in X$, then the such functions are also continuous at this point

$$f(x) \pm g(x), \quad f(x)g(x), \quad f(x)/g(x), \ (g(x_0) \neq 0).$$

5.2. Classification of breakpoints. Examples of the study of functions for continuity

The following types of breaks are distinguished.

Definition of discontinuity of the first kind. *If there are finite one-sided boundaries for the function* $y = f(x)$ *at a point* x_0

$$\lim_{x \to x_0 - 0} f(x) = f(x_0 - 0) \quad \text{and} \quad \lim_{x \to x_0 + 0} f(x) = f(x_0 + 0),$$

moreover, not all numbers $f(x_0-0)$, $f(x_0+0)$, $f(x_0)$ *are equal to each other, then a discontinuity at a point* x_0 *is called a discontinuity of the first kind, and a point* x_0 *is a discontinuity point of the first kind.*

The value $\delta = \left| \lim_{x\to x_0-0} f(x) - \lim_{x\to x_0+0} f(x) \right|$ is called the jump of the function.

Definition of removable discontinuity. *If* $f(x_0-0)=f(x_0+0)\neq f(x_0)$, *then the discontinuity at the point* x_0 *is called removable, and the point* x_0 *is called the point of removable discontinuity.*

Therefore, a removable rupture is a partial case of a rupture of the first kind. In the case of a removable discontinuity, it is enough to determine the function at only one point x_0, putting $f(x_0\pm 0)=f(x_0)$, to obtain a function continuous at the point x_0.

Definition of discontinuity of the second kind. *If at least one of the one-sided boundaries in formula (5.4) does not exist or is equal to infinity, then the discontinuity at a point* x_0 *is called a discontinuity of the second kind, and the-point itself* x_0 *is a discontinuity point of the second kind.*

Example 5.1. Check the continuity of the function

$$f(x)=\begin{cases} -x, \ x<-2; \\ 2^{-x}, \ -2\le x\le 1; \\ 1/2, \ x>1. \end{cases}$$

The solution. The graph of the specified function is shown in Fig. 5.1.

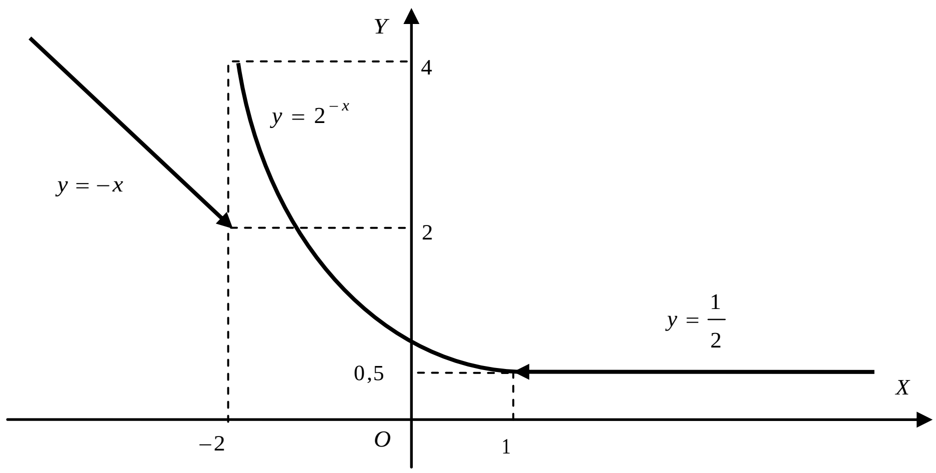

Fig. 5.1. Graph of the function $f(x)$

Let's examine the function at the points suspected of discontinuity, namely, at the points $x = -2$ and $x = 1$, where the branches of the function change their analytical expression. Let's calculate the corresponding one-sided boundaries:

$$\lim_{x\to -2-0} f(x) = \lim_{x\to -2-0} (-x) = 2; \quad \lim_{x\to -2+0} f(x) = \lim_{x\to -2+0} 2^{-x} = 4;$$

$$\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0} 2^{-x} = 1/2; \quad \lim_{x\to 1+0} f(x) = \lim_{x\to 1+0} 1/2 = 1/2; \quad f(1) = 1/2.$$

Therefore, $x = -2$ is the point of discontinuity of the first kind, and $x = 1$ is the point of continuity (Fig. 5.1). The jump of the function at a point $x = -2$ is:

$$\delta = |2 - 4| = 2.$$

Example 5.2. Check the continuity of the function $y(x) = 3^{\frac{1}{x+1}}$.

The solution. It is obvious that is the point $x = -1$ of discontinuity of the given function. To determine the nature of the discontinuity, we will find the limits of the function at this point left and right:

$$\lim_{x\to -1-0} y(x) = \lim_{x\to -1-0} 3^{\frac{1}{x+1}} = 0; \quad \lim_{x\to -1+0} y(x) = \lim_{x\to -1+0} 3^{\frac{1}{x+1}} = +\infty.$$

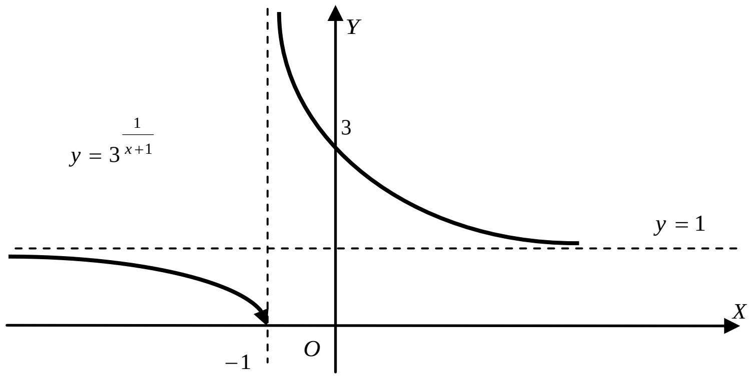

Fig. 5.2. Graph of the function $y(x) = 3^{\frac{1}{x+1}}$

Therefore, the point $x = -1$ is a discontinuity point of the second kind. The graph of the specified function is shown in Fig. 5.2. Here, the straight line $x = -1$ is the vertical asymptote of the curve.

Since $\lim_{x\to \pm\infty} y(x) = 1$, the straight line $y = 1$ is a horizontal asymptote.

Example 5.3. Check the continuity of the function $y(x) = 2^{-\frac{1}{(x-1)^2}}$.

The solution. It is obvious that the point of discontinuity of the given function is the point $x = 1$, since this point does not belong to the domain of the function. To establish the nature of the discontinuity, we will find the one-sided limits of the function at the suspicious point $x = 1$:

$$\lim_{x \to 1-0} y(x) = \lim_{x \to 1-0} 2^{-\frac{1}{(x-1)^2}} = 0; \quad \lim_{x \to 1+0} y(x) = \lim_{x \to 1+0} 2^{-\frac{1}{(x-1)^2}} = 0; \quad \lim_{x \to \pm\infty} y(x) = 1 .$$

The last limit means that the line $y = 1$ is a horizontal asymptote.

It turned out that the two boundaries at the point $x = 1$ are equal, but the function in it is not defined. Therefore, the function has a removable discontinuity (Fig. 5.3), which can be removed if the function is determined by its limit value $y = 0$ at $x \to 1$.

That is, after redefinition of the given function $y(x)$, a new function

$$y_{new}(x) = \begin{cases} 2^{-\frac{1}{(x-1)^2}}, \; if \;\; x \neq 1; \\ 0, \; if \;\; x = 1 \end{cases}$$

will already be continuous on the entire number line $x \in (-\infty; +\infty)$. On the graph (Fig. 5.3), the straight line $x = 1$ is a vertical asymptote, and the point of intersection of the curve $y(x)$ with the axis OY has coordinates $A(0, 1/2)$.

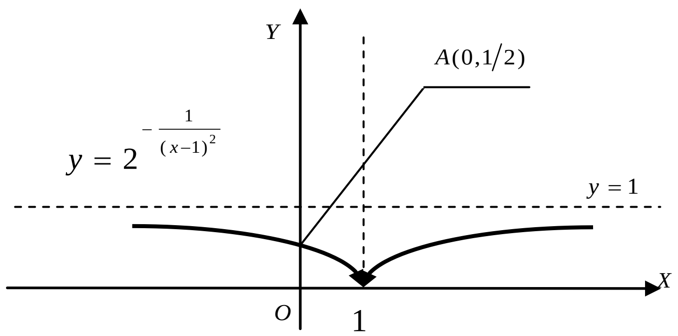

Fig. 5.3. Graph of the function $y(x) = 2^{-\frac{1}{(x-1)^2}}$

5.3. Concept and definition of uniformly continuous functions. Cantor's theorem

Definition of a uniformly continuous function. *A function $f(x)$ is called uniformly continuous on an interval $X = \{x\}$ if for an arbitrary number $\varepsilon > 0$ there exists such a number $\delta(\varepsilon) > 0$ depending only on ε that for any two points x' and x'' from the set $X = \{x\}$ that satisfy the condition $|x' - x''| < \delta(\varepsilon)$, the inequality $|f(x') - f(x'')| < \varepsilon$ holds.*

Remark. The main thing in this definition is that for an arbitrary number $\varepsilon > 0$ there is such a number $\delta(\varepsilon) > 0$ that guarantees the fulfillment of the inequality $|f(x') - f(x'')| < \varepsilon$ at once for all x' and x'' from the set $X = \{x\}$ under the unique condition $|x' - x''| < \delta(\varepsilon)$. In this case, the number δ is dependent ε only on and can be specified before selecting points x' and x'': will be acceptable for all $x \in X$ at the same time.

Uniform continuity means that in all parts of the interval X the same level of closeness of two values of the argument is sufficient to achieve a given level of closeness of the corresponding values of the function.

Note that in a closed interval (segment $[a;b]$), any continuous function will always be uniformly continuous in it, which follows from the following theorem, which belongs to Cantor (G. Cantor).

Cantor's theorem (*on uniform continuity of a function continuous on a segment*). *A function $f(x)$ continuous on a segment $[a;b]$ is uniformly continuous on that segment.*

Example 5.4. Investigate the uniform continuity of the function $f(x) = x + \sin x$ on the entire number line $\mathbb{R}$.

The solution. By the definition of uniform continuity for an arbitrary $\varepsilon > 0$ we have:

$$|f(x') - f(x'')| = |x' - x'' + (\sin x' - \sin x'')| \le |x' - x''| + |\sin x' - \sin x''| =$$

$$= |x' - x''| + 2\left|\sin\frac{x' - x''}{2}\cos\frac{x' + x''}{2}\right| \le |x' - x''| + 2\left|\frac{x' - x''}{2}\right| = 2|x' - x''| < \varepsilon.$$

Here we used estimates $\left|\cos\frac{x' + x''}{2}\right| \le 1$ and $\left|\sin\frac{x' - x''}{2}\right| \le \left|\frac{x' - x''}{2}\right|$.

Therefore, the function $f(x)$ is uniformly continuous on the set $\mathbb{R}$ under the condition $|x' - x''| < \delta = \dfrac{\varepsilon}{2}$.

Example 5.5. Show that the function $f(x) = \sin\dfrac{1}{x}$ is continuous and bounded on the interval $\left(0, {}^{2}\!/_{\pi}\right]$, but is not uniformly continuous on this interval.

The solution. The limitation of the given function $f(x)$ is obvious. The continuity of the composite function $f(x)$ at each point of the interval follows from the fact that it is a superposition of two functions $u = \dfrac{1}{x}$ and $y = \sin u$ continuous on the corresponding intervals.

Let's choose $x_n' = 1/\pi n$ and $x_n'' = {}^{2}\!\big/_{\pi(2n+1)}$, where n is an arbitrary natural number. Then the values of the function at the selected points will be as follows:

$$f(x_n') = \sin n\pi = 0\,,\quad f(x_n'') = \sin\left({}^{(2n+1)\pi}\!\big/_{2}\right) = \pm 1\,.$$

Therefore, we have: $\left|f(x_n') - f(x_n'')\right| = 1 \geq \varepsilon,\ \forall\,\varepsilon \in (0,1]$ despite the fact that as the number n increases, the modulus of the difference $\left|x_n' - x_n''\right| = {}^{1}\!\big/_{n(2n+1)\pi}$ can be arbitrarily small. Here, with $\varepsilon = 1$, it is impossible to find a number δ that would be acceptable simultaneously for all points $x_n', x_n'' \in \left(0, {}^{2}\!\big/_{\pi}\right]$ in the sense of fulfilling the inequality $\left|f(x_n') - f(x_n'')\right| \leq \varepsilon$, although for each individual value of x (due to the continuity of the function), such a number δ exists.

Example 5.6. Investigate for uniform continuity the function

$$f(x) = \frac{x}{9 - x^2},\quad x \in [-2, 2]\,.$$

The solution. The function $f(x)$ is continuous on the segment $[-2, 2]$, and therefore, according to Cantor's theorem, $f(x)$ is uniformly continuous on this segment.

Tasks for classroom and independent work

I. Find the left and right boundaries of the functions $f(x)$ at the point x_0.

5.1. $f(x) = \dfrac{1}{1 + 2^{\frac{1}{x-1}}},\quad x_0 = 1\,.$

5.2. $f(x) = \dfrac{2}{x - 3},\quad x_0 = 3.$

5.3. $f(x) = \dfrac{|x - 1|}{x - 1},\quad x_0 = 1\,.$

5.4. $f(x) = 3^{\frac{1}{x-2}},\quad x_0 = 2\,.$

5.5. $f(x) = 2^{\frac{1}{(x-2)^2}},\quad x_0 = 2\,.$

5.6. $f(x) = \dfrac{1}{1 + e^{\frac{1}{x}}},\quad x_0 = 0\,.$

5.7. $f(x) = \begin{cases} 2x - 1, & x < 2 \\ x^2, & x \geq 2 \end{cases},\quad x_0 = 2\,.$

5.8. The function is defined as follows:

$$y(x) = \begin{cases} x + 1,\ if\ \ x \leq 1; \\ 3 - ax^2,\ if\ \ x > 1\,. \end{cases}$$

At what value of parameter a will the function be continuous? Construct its graph.

5.9. The function $y = \dfrac{4}{1+2^{\frac{1}{(x-1)^2}}}$ is not defined at point $x = 1$. Can it be determined at a point $x = 1$ so that the new function $y_{new}(x)$ is continuous at this point? Plot the graph of this new function.

5.10. Investigate the nature of the discontinuity of the function $y(x) = 2^{-2^{\frac{1}{1-x}}}$ at the point $x = 1$. Is it possible to determine the function $y(x)$ at $x = 1$ in such a way that the new function $y_{new}(x)$ turns out to be continuous at point $x = 1$?

II. Investigate given functions for continuity and classify their discontinuity points.

5.11. $y = 3^{\frac{1}{4-x}}$.

5.12. $y = \dfrac{2}{1+2^{\frac{1}{x-2}}}$.

5.13. $y = \operatorname{arctg}\dfrac{1}{x}$.

5.14. $y = \dfrac{2^{\frac{1}{x}}-1}{1+2^{\frac{1}{x}}}$.

5.15. $y = \dfrac{1}{\ln x}$.

5.16. $y = \cos^2\dfrac{1}{x}$.

5.17. $y(x) = \sqrt{\dfrac{1-\cos\pi x}{4-x^2}}$.

5.18. $y = \dfrac{1}{1-e^{\frac{x}{1-x}}}$.

5.19. $y(x) = \begin{cases} x\cdot\sin(\pi/x), & x \neq 0; \\ 0, & x = 0. \end{cases}$

5.20. $y(x) = \begin{cases} \cos(\pi x/2), & |x| \le 1; \\ |x-1|, & |x| > 1. \end{cases}$

III. Investigate the given functions for continuity and plot their graphs.

5.21. $y = e^{-\frac{1}{x}}$.

5.22. $y = 1 - e^{-\frac{1}{x^2}}$.

5.23. $f(x) = \begin{cases} 2tg\dfrac{x}{2}, & |x| < \dfrac{\pi}{2}, \\ 2, & |x| \ge \dfrac{\pi}{2}; \end{cases}$

5.24. $f(x) = \begin{cases} 4\cos\left(x-\dfrac{\pi}{3}\right), & x < \dfrac{\pi}{3}, \\ 4, & x \ge \dfrac{\pi}{3}. \end{cases}$

5.25. $y = \lim\limits_{n\to\infty}\dfrac{1}{1+x^n}$, $x \ge 0$.

5.26. $y = \lim\limits_{n\to\infty}\sqrt[n]{1+x^{2n}}$.

5.27. $y = \lim\limits_{n\to\infty}\dfrac{x+x^2e^{nx}}{1+x^{nx}}$.

5.28. $y(x) = \lim\limits_{t\to+\infty}\dfrac{\ln(1+e^{xt})}{\ln(1+e^t)}$.

IV. The function $f(x)$ is not defined at point $x=0$. Determine the number $f(0)$ so that $f(x)$ is continuous at $x=0$.

5.29. $f(x)=\dfrac{\sqrt{1+x}-1}{\sqrt[3]{1+x}-1}$.

5.30. $f(x)=\sin x\sin\dfrac{1}{x}$.

5.31. $f(x)=x\ln^2 x$.

5.32. $f(x)=(1+x)^{\frac{1}{x}}$.

V. Check for uniform continuity of the function on the specified interval.

5.33. $f(x)=\ln x,\ x\in(0,1)$.

5.34. $f(x)=\dfrac{\sin x}{x},\ x\in(0,\pi)$.

5.35. $f(x)=x\sin x,\ x\in[0,+\infty)$.

5.36. $f(x)=\sin x,\ x\in(-\infty,+\infty)$.

5.37. $f(x)=\operatorname{arctg} x,\ x\in(-\infty,+\infty)$.

5.38. $f(x)=\dfrac{1}{x},\ 1)\ x\in(0,1/10),\ 2)\ x\in[1/10,1]$.

5.39. $f(x)=\sqrt{x},\ 1)\ x\in[1,+\infty),\ 2)\ x\in[0,+\infty)$.

5.40. Show that the function $f(x)=\sin\dfrac{\pi}{x}$ is continuous and bounded on the interval $(0,1)$, but is not uniformly continuous on this interval.

5.41. Will the function $y=\dfrac{1}{2-x}$ be uniformly continuous on the interval $[0,2)$? And on the interval $[0,2-\delta]$, where δ is an arbitrary fixed positive number?

5.42. Will the function $y=\sin(x^2)$ be uniformly continuous on the interval $(-\infty,+\infty)$? And on the interval $\left(\sqrt{\dfrac{\pi}{6}},\sqrt{\dfrac{\pi}{2}}\right)$?

Answers and instructions

5.1. $f(1-0)=1,\ f(1+0)=0$.

5.2. $f(3-0)=-\infty,\ f(3+0)=+\infty$.

5.3. $f(1-0)=-1,\ f(1+0)=1$.

5.4. $f(2-0)=0,\ f(2+0)=\infty$.

5.5. $f(2\pm 0)=\infty$.

5.6. $f(-0)=1,\ f(+0)=0$.

5.7. $f(2-0)=3,\ f(2+0)=4$.

5.8. $a=1$.

5.9. $y(1)=0$.

5.10. $x=1$ – breaking point of the 1st kind.

5.11. $x=4$ – breakpoint of the 2nd kind.

5.12. $x=2$ – breaking point of the 1st kind.

5.13. $x=0$ – breaking point of the 1st kind.

5.14. $x=0$ – breaking point of the 1st kind.

5.15. $x=0$ – point of removable rupture; $x=1$ – breakpoint of the 2nd kind.

5.16. $x = 0$ – breakpoint of the 2nd kind.
5.17. $x = \pm 2$ – removable break points.
5.18. $x = 0,\ x = 1$ – breaking points of the 2nd kind.
5.19. Continuous. **5.20.** $x = -1$ – breaking point of the 1st kind.
5.21. $x = 0$ – breakpoint of the 2nd kind.
5.22. $x = 0$ – the point of removable rupture.
5.23. $x = -\pi/2$ – breaking point of the 1st kind. **5.24.** Continuous.
5.25. $y = 1$, if $x \in [0;1)$; $y = 1/2$, if $x = 1$; $y = 0$, if $x > 1$; $x = 1$ – breaking point of the 1st kind.
5.26. $y = 1$, if $|x| \le 1$; $y = x^2$, if $|x| > 1$; the function is continuous.
5.27. $y = x$, if $x \le 0$; $y = x^2$, if $x > 0$; the function is continuous.
5.28. $y = 0$, if $x \le 0$; $y = x$, if $x > 0$; the function is continuous.
5.29. $f(0) = 1{,}5$. **5.30.** $f(0) = 0$. **5.31.** $f(0) = 0$.
5.32. $f(0) = e$. **5.33.** Is not uniformly continuous.
5.34. Uniformly continuous. **5.35.** Is not uniformly continuous.
5.36. Uniformly continuous. **5.37.** Uniformly continuous.
5.38. 1) Is not uniformly continuous; 2) uniformly continuous.
5.39. 1) Uniformly continuous; 2) uniformly continuous. **5.42.** No. Yes.

Section II. Differential calculus of functions of one variable (FOV)

PRACTICAL LESSON 6

Derivative. Differentiation rules and a table of derivatives. Application of the derivative. Differentiation technique

6.1. The concept of derivative and its different notations. Differentiation rules and a table of derivatives

We will briefly state the theoretical provisions related to the subject of this practical lesson.

Definition 6.1. *If there is a limit of the ratio of the increment* Δy *of the function* $y = f(x)$ *to the increment of the argument* Δx, *then* $\Delta x \to 0$, *it is called the derivative of the function* $y = f(x)$ *at a point* x.

The derivative of a function $y = f(x)$ at a point x is denoted by one of the following symbols:

$$y'; \quad \frac{dy}{dx}; \quad \frac{df}{dx}; \quad y'_x; \quad f'(x).$$

So, by definition, we have:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{6.1}$$

Let's formulate the algorithm for finding the derivative according to definition (6.1). To find the derivative of a function $y = f(x)$ at some point x, you need:

a) give the value x an arbitrary increment Δx and find the corresponding increment of the function

$$\Delta y = f(x + \Delta x) - f(x);$$

b) find a relationship:

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x};$$

c) find the limit of this ratio:

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

If this limit exists, then it is equal to the derivative $f'(x)$.

Differentiation rules

Let be $u = u(x),\ v = v(x)$ - differentiable functions, and C – be a constant value. We will present the formulas used to find derivatives:

1. $C' = 0$.
2. $(u \pm v)' = u' \pm v'$.
3. $(uv)' = u'v + uv'$.
4. $(Cu)' = Cu'$.
5. $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2},\ (v(x) \neq 0)$.

Table of derivatives

In practice, it is often necessary to calculate the derivatives of composite functions, therefore, in the given formulas (Table 6.1), instead of the independent argument x, we will consider the dependent argument u, where $u = \varphi(x)$.

Table 6.1

1. $\left(u^{\alpha}\right)' = \alpha u^{\alpha-1} u'$, ***in particular*** $\left(\sqrt{u}\right)' = \frac{1}{2\sqrt{u}} u'$, $\left(\frac{1}{u}\right)' = -\frac{1}{u^2} u'$.	
2. $\left(a^{u}\right)' = a^{u} \ln a\ u'$, ***in particular*** $\left(e^{u}\right)' = e^{u} u'$.	
3. $(\log_a u)' = \frac{1}{u \ln a} u'$, ***in particular*** $(\ln u)' = \frac{1}{u} u'$.	
4. $(\sin u)' = \cos u\ u'$.	**5.** $(\cos u)' = -\sin u\ u'$.
6. $(\operatorname{tg} u)' = \frac{1}{\cos^2 u} u'$.	**7.** $(\operatorname{ctg} u)' = -\frac{1}{\sin^2 u} u'$.
8. $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} u'$.	**9.** $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} u'$.
10. $(\operatorname{arctg} u)' = \frac{1}{1+u^2} u'$.	**11.** $(\operatorname{arcctg} u)' = -\frac{1}{1+u^2} u'$.
12. $(shu)' = chu \cdot u'$.	**13.** $(chu)' = shu \cdot u'$.
14. $(thu)' = \frac{1}{ch^2 u} \cdot u'$.	**15.** $(cthu)' = -\frac{1}{sh^2 u} \cdot u'$.

6.2. Application of the derivative in mechanics and geometry

1. **The mechanical content of the derivative.** Let the material point move in a straight line according to the law $S = S(t)$, where t – is time and $S(t)$ – is the path traveled by the material point during this time. Then ***the instantaneous speed at a given moment of time*** is a derivative of the law $S = S(t)$ of motion of a material point in time t: $V = \frac{dS}{dt} = \dot{S}(t)$. If the speed becomes zero ($\dot{S}(t) = 0$), then the material point stops; if it has a negative sign ($\dot{S}(t) < 0$), then the point moves in the opposite direction. This is the ***mechanical content of the derivative***.

Remark. For the time derivative, a special notation is used in the form of a dot above the function – $\dot{S}(t)$.

Generalizing this concept to other ***physical processes***, we can say: if a function $y = f(x)$ describes some physical process, then the derivative $y' = f'(x)$ is ***the rate of change of this process***. In other words, no matter what physical meaning the functional dependence $y = f(x)$ reflects, the ratio of increments $\frac{\Delta y}{\Delta x}$ can be considered as ***the average rate*** of change of the function y relative to the argument x, and the derivative $f'(x)$ – as ***the instantaneous rate*** of change of the function. For example, it is known that the volume V of a body depends on the temperature T of the external environment in which this body is located, that is, there is a dependence $V = V(T)$. Then the derivative $V'(T)$ determines the instantaneous rate of change of the volume of the body depending on the change in temperature of the surrounding medium. This is ***the physical meaning of the derivative***.

An example of the application of the derivative in mechanics. A material point moves in a straight line along the horizontal axis according to the law $f(t) = (t^2 + 1)e^{-t}$. Find all moments of time at which the point stops.

The solution. The condition of stopping the material point will be fulfilled when the instantaneous velocity $v(t)$ is equal to zero. Let's find the derivative of the given function:

$$v(t) = f'(t) = \left[(t^2 + 1)e^{-t}\right]' = -(t-1)^2 e^{-t}.$$

Let's equate the instantaneous speed to zero

$$v(t) = 0 \Rightarrow -(t-1)^2 e^{-t} \to 0 \Rightarrow t_1 = 1, t_2 \to +\infty .$$

Fig. 6.1 shows the graph of the function $v(t)$ for illustration.

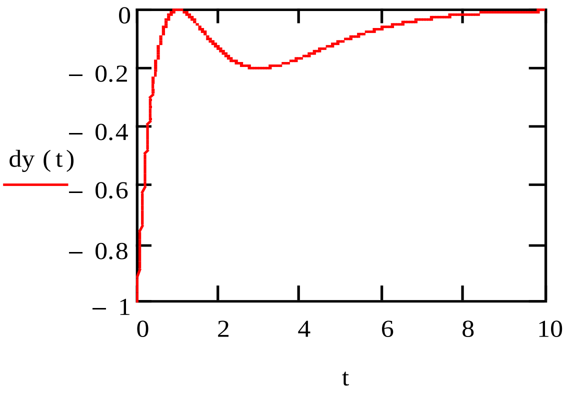

Fig. 6.1

***Conclusion*:** we have only two points in time when a material point has stops: an instantaneous stop under the condition $t_1 = 1$ (when the curve $f(t)$ has an inflection) and a final stop under the condition $t_2 \to +\infty$ (when the point finally stops).

2. **The geometric content of the derivative.** The derivative $f'(x_0)$ at some point $M_0(x_0; y_0)$ of the curve is equal to ***the angular coefficient of the tangent*** to the curve $y = f(x)$ or the tangent of the angle α (Fig. 6.2), which forms the tangent to the curve at this point $M_0(x_0; y_0)$ with the positive direction of the axis $OX: k = tg\alpha = f'(x_0)$. This is ***the geometric content of the derivative***.

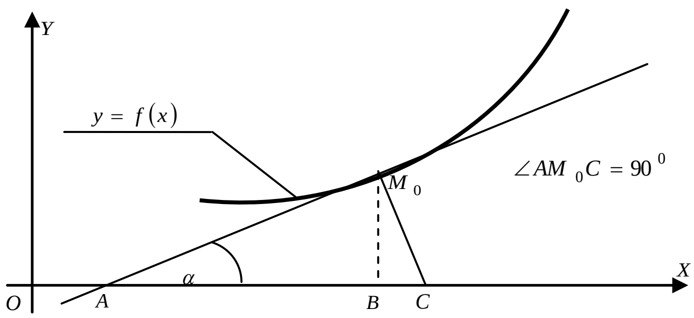

Fig. 6.2

Let's find the equation of the tangent to the curve $y = f(x)$. Since the tangent passes through the point $M_0(x_0; y_0)$ in the direction determined by the angle α, then, putting $k = f'(x_0)$ in the formula $y - y_0 = k(x - x_0)$ for a straight line, we get:

$$y - y_0 = f'(x_0)(x - x_0). \quad (6.2)$$

Equation (6.2) is called ***the equation of the tangent to the curve*** $y = f(x)$ at the point $M_0(x_0; y_0)$. In particular, if the function at a point $M_0(x_0; y_0)$ has an infinite derivative, then the tangent at this point is parallel to the axis OY, and its equation is: $x = x_0$.

Example 6.1. Find the tangent to the curve $y = x^3$ at the point $x_0 = 1$.

The solution. Let's use formula (6.2). To do this, we will find the value of the derivative of the function $y = x^3$ at the point $x_0 = 1$: $f'(x)\big|_{x=1} = 3x^2\big|_{x=1} = 3$.

Now let's write the equation of the tangent at the point $M_0(1;1)$:

$$y - 1 = 3(x - 1) \text{ or } y = 3x - 2.$$

Definition 6.2. *The normal to the curve* $y = f(x)$ *is the straight line passing through the point of contact* $M_0(x_0; y_0)$ *perpendicular to the tangent at the same point.*

Since the angular coefficients of the tangent k_D and the normal k_N are related to each other by the corresponding condition of the perpendicularity $1 + k_D \cdot k_N = 0$ of two straight lines, the equation of the normal to the curve $y = f(x)$ at the point $M_0(x_0; y_0)$ has the form:

$$y - y_0 = -\frac{1}{f'(x_0)}(x - x_0). \tag{6.3}$$

Using equation (6.2), you can find the length of the segment AM_0 (Fig. 6.2), which is called ***the length of the tangent segment*** (as the distance between the points A and M_0), and the length of the segment AB, which is called ***the subtangent*** (as the distance between the points A and B). Similarly, with the help of equation (6.3), ***the length of the segment of the normal*** M_0C and ***subnormal*** BC is found.

6.3. The technique of differentiation of functions by definition. Differentiation of a composite function

Let's consider some examples on the topic of finding the derivative according to definition (6.1).

Example 6.2. Find the derivative of a power function $y = x^\alpha$, where α – is an arbitrary real number.

The solution. We apply the equivalence $(1+z)^{\alpha}-1\sim\alpha\cdot z$ under the condition $z\to 0$. As a result, we get:

$$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^{\alpha}-x^{\alpha}}{\Delta x}=x^{\alpha}\lim_{\Delta x\to 0}\frac{\left(1+\frac{\Delta x}{x}\right)^{\alpha}-1}{\Delta x}=$$

$$=x^{\alpha}\lim_{\Delta x\to 0}\frac{\alpha\frac{\Delta x}{x}}{\Delta x}=\alpha x^{\alpha}\frac{1}{x}=\alpha x^{\alpha-1}.$$

So, we got the tabular formula: $\left(x^{\alpha}\right)'=\alpha x^{\alpha-1}$.

Example 6.3. Find the derivative of a function $y(x)=\sin x$.

The solution. Here we apply the first important limit:

$$(\sin x)'=\lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}=\lim_{\Delta x\to 0}\frac{2\sin\left(\frac{\Delta x}{2}\right)\cos\left(x+\frac{\Delta x}{2}\right)}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{\sin\left(\frac{\Delta x}{2}\right)}{\frac{\Delta x}{2}}\lim_{\Delta x\to 0}\cos\left(x+\frac{\Delta x}{2}\right)=\cos x.$$

So, we got the tabular formula: $(\sin x)'=\cos x$. The formula is proved similarly $(\cos x)'=-\sin x$.

Example 6.4. Find the derivative of an exponential function $y=a^{x}\,(a>0,\,a\neq 1)$.

The solution. Let's use equivalence $a^{z}-1\sim z\ln a$ under the condition $z\to 0$:

$$\left(a^{x}\right)'=\lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^{x}}{\Delta x}=a^{x}\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}=a^{x}\lim_{\Delta x\to 0}\frac{\Delta x\ln a}{\Delta x}=a^{x}\ln a.$$

So, we got the tabular formula: $\left(a^{x}\right)'=a^{x}\ln a$. The tabular formula is proved similarly $(\log_a x)'=\log_a e/x$.

Consider the differentiation of ***the product of two functions***.

Example 6.5. Find the derivative of a function: $y=\left(1-x^{2}\right)\arcsin x$.

The solution. Using the rule of differentiation of the product of functions $(uv)' = u'v + uv'$, we get:

$$y' = \left(1 - x^2\right)' \arcsin x + \left(1 - x^2\right)\left(\arcsin x\right)' = -2x \arcsin x + \left(1 - x^2\right)\left(1 - x^2\right)^{-\frac{1}{2}} =$$

$$= \sqrt{1 - x^2} - 2x \arcsin x.$$

Consider the differentiation of the fractions u/v of two functions.

Example 6.6. Using the rule of differentiation of fractions of functions $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, we obtain for the functions $\operatorname{tg} x$ and $\operatorname{ctg} x$:

$$(\operatorname{tg} x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x};$$

$$(\operatorname{ctg} x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \sin x - \cos x(\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}.$$

We will show how to differentiate a ***composite*** function.

Example 6.7. Find the derivative of a function: $y = \operatorname{arctg}^5 x$.

The solution. The given function is composed of: $y = u^5$, where $u = \operatorname{arctg} x$. Therefore

$$y'_x = y'_u \cdot u'_x = 5u^4 \cdot \frac{1}{1 + x^2} = \frac{5 \operatorname{arctg}^4 x}{1 + x^2}.$$

Example 6.8. Find the derivative of a composite function: $y(x) = \ln\left(\operatorname{tg}\left(x^2\right)\right)$.

The solution.

$$\left[\ln\left(\operatorname{tg}\left(x^2\right)\right)\right]' = \frac{1}{\operatorname{tg} x^2}\left(\operatorname{tg} x^2\right)' = \frac{1}{\operatorname{tg}(x^2)} \frac{1}{\cos^2(x^2)}\left(x^2\right)' = \frac{4x}{\sin(2x^2)}.$$

6.4. Differentiation of a parametrically defined function and an implicitly defined function. Logarithmic differentiation

Consider the differentiation of ***parametrically specified functions***.

Example 6.9. Write the equation of the tangent to the cycloid $x = 2(t - \sin t)$, $y = 2(1 - \cos t)$ at the point $M_0(x_0; y_0)$ corresponding to the parameter $t_0 = \pi/2$.

The solution. Let's find the coordinates of the point $M_0(x_0; y_0)$

$$x_0 = 2(t - \sin t)\Big|_{t=\frac{\pi}{2}} = \pi - 2; \quad y_0 = 2(1 - \cos t)\Big|_{t=\frac{\pi}{2}} = 2.$$

Let's calculate the value of the derivative y'_x of $t_0 = \dfrac{\pi}{2}$:

$$y'_x\Big|_{t=\frac{\pi}{2}} = \left(\frac{2\sin t}{2(1 - \cos t)}\right)\Bigg|_{t=\frac{\pi}{2}} = 1.$$

Let's make the equation of the desired tangent:

$$y - 2 = 1 \cdot (x - (\pi - 2)) \text{ or } y = x + 4 - \pi.$$

Let us consider two examples of differentiation of an implicitly given function.

Example 6.10. Find the derivative y' if

$$F(x, y) = x^2 + y^2 - \operatorname{arctg} y + 3x - 2y - 2 = 0.$$

The solution. Let's find the derivative of both parts of the given equation, considering y the function x. As a result, we have:

$$2x + 2yy' - \frac{y'}{1 + y^2} + 3 - 2y' = 0,$$

$$y'\left(\frac{1}{1 + y^2} - 2y + 2\right) = 2x + 3 \Rightarrow y' = \frac{(2x + 3)(1 + y^2)}{3 - 2y(y^2 - y + 1)}.$$

Example 6.11. Find the derivative of the implicit function y if $2^x + 2^y = 2^{x+y}$.

The solution. Since $\left(2^y\right)' = 2^y \ln 2 \cdot y'$, $\left(2^{x+y}\right)' = 2^{x+y} \ln 2 \cdot (1 + y')$, then after differentiating the given equation we get:

$$2^x \ln 2 + 2^y \ln 2 \cdot y' = 2^{x+y} \ln 2 \cdot (1 + y') \Rightarrow y'\left(2^y - 2^{x+y}\right) = 2^{x+y} - 2^x.$$

Finally we have: $y' = \dfrac{2^{x+y} - 2^x}{2^y - 2^{x+y}} = -2^{y-x}$; $y' = 1 - 2^y$; $y' = (1 - 2^x)^{-1}$.

Let's explain the technique of ***logarithmic differentiation***.

Example 6.12. Find the derivative of the function $y = \sqrt[3]{x^{-5}(1+x)^4}$.

The solution. This function can be differentiated by the rule of fraction differentiation. However, this method is very cumbersome. Let's apply logarithmic differentiation. After logarithmizing the function, we get:

$$\ln y = \frac{1}{3}\left[4\ln(1+x) - 5\ln x\right].$$

After differentiation, we get: $\frac{y'}{y} = \frac{1}{3}\left(\frac{4}{1+x} - \frac{5}{x}\right) = -\frac{x+5}{3x(1+x)}$.

And finally we have:

$$y' = y(\ln y)' = -\frac{x+5}{3x(1+x)}\sqrt[3]{\frac{(x+1)^4}{x^5}} = -\frac{x+5}{3x^2}\sqrt[3]{\frac{x+1}{x^2}}.$$

Another example of a function whose derivative is easier to find by logarithmic differentiation is an ***exponent-power function***

$$y = u(x)^{v(x)}, \tag{6.4}$$

where u, v − given and differentiable functions from x.

Let's find the derivative of function (6.4) in the general form:

$$\ln y = v\ln u\,;\ \frac{y'}{y} = v'\ln u + v\frac{u'}{u};\ y' = u^v\left(v'\ln u + v\frac{u'}{u}\right) = u^v\ln u\cdot v' + vu^{v-1}u'.$$

Therefore, the derivative of the exponential-power function (6.4) is equal to the sum of the derivative of the exponential function under the condition that $u = const$, and the derivative of the power function under the condition that $v = const$:

$$\left(u^v\right)' = u^v\ln u\cdot v' + vu^{v-1}u'. \tag{6.5}$$

Remark. The derivative of the exponential-power function (6.4) can be found in another way, using the identity $u^v = e^{\ln u^v} = e^{v\ln u}$. Indeed, after its differentiation we will get

$$\left(u^v\right)' = \left(e^{v\ln u}\right)' = u^v(v\ln u)'. \tag{6.6}$$

Example 6.13. Find the equation of the tangent to the curve $y = x^{x^x}$ at the point $x = e$.

The solution. To calculate the derivative, we will use logarithmic differentiation:

$$y' = x^{x^x}\left(x^x \ln x\right)' = x^{x^x}\left((x^x)'\ln x + x^{x-1}\right).$$

Now we will separately find the derivative of the function x^x:

$$(x^x)' = x^x(x\ln x)' = x^x(\ln x + 1).$$

Thus, the derivative y' of the given function will have the following form:

$$y' = x^{x^x}\left(x^x(\ln x + 1)\ln x + x^{x-1}\right) = x^{x^x + x}\left((\ln x + 1)\ln x + \frac{1}{x}\right).$$

Now let's calculate the value of the derivative y' at the point $x = e$:

$$y'(e) = e^{e+e^e}\left(2 + \frac{1}{e}\right).$$

The equation of the desired tangent has the following form:

$$y - e^{e^e} = e^{e+e^e}\left(2 + \frac{1}{e}\right)\cdot\left(x - e\right).$$

Example 6.14. Find the derivative y' of the implicit function: $x^y = y^x$.

The solution. The given function is an ***exponential-power*** function, which is specified in an ***implicit form***. Let's apply logarithmic differentiation. As a result, we get:

$$y\ln x = x\ln y \Rightarrow y'\ln x + {y}/{x} = \ln y + {xy'}/{y} \Rightarrow$$

$$\Rightarrow y'\left(\ln x - {x}/{y}\right) = \ln y - {y}/{x} \Rightarrow y' = \frac{y(x\ln y - y)}{x(y\ln x - x)} = \left(\frac{y}{x}\right)^2\cdot\frac{\ln x - 1}{\ln y - 1}.$$

6.5. Research on differentiability of functions

Example 6.15. Find the derivative y' of the function $y = |x|,\ x \neq 0$.

The solution. Using the definition of the derivative for a given function, we write the expressions for the right and left derivatives, respectively.

Let $\Delta x \to +0$, then ***the right*** derivative has the form:

$$y' = \lim_{\Delta x \to +0}\frac{y(0 + \Delta x) - y(0)}{\Delta x} = \lim_{\Delta x \to +0}\frac{\Delta x}{\Delta x} = 1.$$

Let $\Delta x \to -0$, then ***the left*** derivative has the form:

$$y' = \lim_{\Delta x \to -0}\frac{y(0 + \Delta x) - y(0)}{\Delta x} = \lim_{\Delta x \to -0}\frac{y(0 + \Delta x) - 0}{\Delta x} = \left|\Delta x = -\Delta z\right| = .$$

$$= \lim_{\Delta z \to +0} \frac{y(0 - \Delta z)}{-\Delta z} = \lim_{\Delta z \to +0} \frac{|0 - \Delta z|}{-\Delta z} = \lim_{\Delta x \to -0} \frac{\Delta z}{-\Delta z} = -1 .$$

At the end of the solution, we note that this function does not have a derivative at the point $x = 0$, since it is a “corner” point.

Example 6.16. Find the derivative $y'(x)$ of a function:

$$y(x) = \begin{cases} \dfrac{\sqrt{1 + x^2} - 1}{x}, if \;\; x \neq 0 \\ 0, if \;\; x = 0 \end{cases} .$$

Is the given function $y(x)$ continuous? Is its derivative $y'(x)$ a continuous function?

The solution. In this example, we will also use the definition of a derivative. First, we will find the derivative of the given function under the condition that $x \neq 0$.

Therefore, if $x \neq 0$, then:

$$\left(\frac{\sqrt{1 + x^2} - 1}{x} \right)' = \frac{\sqrt{1 + x^2} - 1}{x^2 \sqrt{1 + x^2}} .$$

Now, by definition, we will find the derivative at a special point $x = 0$:

$$y' = \lim_{x \to 0} \frac{y(x) - 0}{x} = \lim_{x \to 0} \frac{\dfrac{\sqrt{1 + x^2} - 1}{x} - 0}{x} = \frac{1}{2} .$$

Let’s determine whether the given function is continuous at a given point $x = 0$. Let’s find the limit of the upper branch of the function at a point $x = 0$:

$$\lim_{x \to 0} \frac{\sqrt{1 + x^2} - 1}{x} = 0 .$$

We see that the value of the limit coincides with the value of the function at the point beyond the second branch. Hence we conclude that the given function is continuous at the point $x = 0$. Now we will find out whether the derivative of the given function is continuous at the same point. For this, it is necessary to find the limit of the derivative for the first branch at the point $x = 0$, i.e.:

$$\lim_{x \to 0} \frac{\sqrt{1 + x^2} - 1}{x^2 \sqrt{1 + x^2}} = \lim_{x \to 0} \frac{\frac{1}{2} x^2}{x^2 \sqrt{1 + x^2}} = \frac{1}{2} .$$

Again, we can state that the derivative of the given function is also a continuous function, since this limit coincides with the value of the derivative at the point $x = 0$.

Conclusion**:** we have only two points in time when a material point has stops: an instantaneous stop under the condition $t_1 = 1$ (when the curve $f(t)$ has an inflection) and a final stop under the condition $t_2 \to +\infty$ (when the point finally stops).

Tasks for classroom and independent work

6.1. Find the left and right derivatives of the function $f(x) = e^{-|x|}$ at the point $x = 0$. Is the function $f(x)$ continuous at this point? Is it differentiable at this point?

6.2. Prove by definition the formula for differentiation of inverse trigonometric functions $y(x) = \arcsin x$, $y(x) = \operatorname{arctg} x$.

I. Find derivatives of functions:

6.3. $y = \operatorname{arctg}^2(\cos x) + \sqrt{\arcsin(\ln(\sin x))}$.

6.4. $y = \sqrt[3]{\dfrac{1+x^3}{1-x^3}}$.

6.5. $y = \sin(\cos^2 x) \cdot \cos(\sin^2 x)$.

6.6. $y = \dfrac{\cos x}{2\sin^2 x}$.

6.7. $y = \dfrac{\sin x - x\cos x}{\cos x + x\sin x}$.

6.8. $y = 4\sqrt[3]{ctg^2 x} + \sqrt[3]{ctg^8 x}$.

6.9. $y = e^x\left(1 + ctg\dfrac{x}{2}\right)$.

6.10. $y = \ln(\ln^2(\ln^3 x))$.

6.11. $y = \dfrac{1}{4}\ln\dfrac{x^2-1}{x^2+1}$.

6.12. $y = \dfrac{1}{2}ctg^2 x + \ln\sin x$.

6.13. $y = \ln\left(\operatorname{tg}\dfrac{x}{2}\right) - \cos x \cdot \ln(\operatorname{tg} x)$.

6.14. $y = \arccos\sqrt{1-x^2}$.

6.15. $y = \operatorname{arctg}\left(\dfrac{1+x}{1-x}\right)$.

6.16. $y = \operatorname{arctg}\sqrt{x^2-1} - \dfrac{\ln x}{\sqrt{x^2-1}}$.

6.17. $y = \dfrac{\arcsin x}{\sqrt{1-x^2}} + \dfrac{1}{2}\ln\dfrac{1-x}{1+x}$.

6.18. $y = \ln(e^x + \sqrt{1+e^{2x}})$.

6.19. $y = \ln(chx) + \dfrac{1}{2ch^2 x}$.

6.20. $y = \operatorname{arctg}(\operatorname{th} x)$.

6.21. Find $y'\left(\pi/2\right)$, if $y(x) = 1/2\, \operatorname{tg}^2(\sin x) + \ln(\cos(\sin x))$.

6.22. Find $y'(1)$, if $y(x) = x + (x-1)\arcsin\sqrt{\frac{x}{x+1}}$.

II. Find derivatives y'_x of functions given implicitly:

6.23. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

6.24. $\operatorname{arctg}\frac{y}{x} = \ln\sqrt{x^2 + y^2}$.

6.25. $xy = \arcsin(x + y)$.

6.26. $\lg(x + y) - \frac{x}{y} = 10$.

6.27. What is the essence of ***logarithmic differentiation***? Prove the formulas by the method of logarithmic differentiation: $\left(x^p\right)' = px^{p-1}$ and $\left(a^x\right)' = a^x \ln a$.

6.28. Differentiate the function $y = \log_u v$, where $u = u(x)$ and $v = v(x)$. Using the results of this task, find the equation of the tangent to the curve $y = \log_{\sin x}(\cos x)$ at the point $x = \frac{\pi}{4}$.

III. Find the derivative of a function $y = f(x)$ by logarithmic differentiation:

6.29. $y = \sqrt[x]{x},\ x > 0$.

6.30. $y = \left(\frac{x}{x+1}\right)^x$.

6.31. $y = (\cos 2x)^{tgx}$.

6.32. $y = (\sin 8x)^{\cos 5x}$.

6.33. $y = \sqrt[7]{\frac{e^{\sin 4x}(x^2 + 6x + 10)}{\operatorname{arctg}^2\left(\frac{1}{x}\right)\cdot \operatorname{sh}^6 2x}}$.

6.34. $y = \frac{x^2}{1-x}\sqrt[3]{\frac{3-x}{(3+x)^2}}$.

IV. Find the derivative y'_x of a function $y = f(x)$ given parametrically:

6.35. $x = \sin^2 t,\ y = \cos^2 t$.

6.36. $x = a\operatorname{ch} t,\ y = b\operatorname{sh} t$.

6.37. $x = e^{t^2},\ y = t^3$.

6.38. $x = t + \ln\sin t,\ y = t + \ln\cos t$.

6.39. Let $x(t) = e^{-t}\sin t,\ y(t) = e^{t}\cos t$. Make sure that $y'_x(1) = e^2$.

6.40. Let $x^{y^2} + y^2\ln x - 4 = 0$. Make sure that $y'(e) = -\frac{y}{2e}$.

6.41. At what angles do the lines intersect $y^2 = x$ and $y = x^2$ at their intersection points?

6.42. At what angles do the lines $y = e^{-3x}$ and $y = e^{2x^2+1}$ intersect?

V. Write the equation of the tangent and the normal to the curves at the given points:

6.43. $\frac{x^2}{100} + \frac{y^2}{64} = 1,\ M_0(6;\ 6,4)$.

6.44. $xy + \ln y = 1,\ M_0(1;\ 1)$.

6.45. $x = 2t - t^2,\ y = 3t - t^3,\ t_0 = 0$. **6.46.** $x = \dfrac{2t + t^2}{1 + t^3},\ y = \dfrac{2t - t^2}{1 + t^3},\ t_0 = 1$.

6.47. Write the equation of the normal to the hyperbola $y = \dfrac{x-1}{x+2}$ parallel to the line $9x + 3y + 2 = 0$.

6.48. To the line $y = x^4 + 6x + 6$, write the equation of the tangent perpendicular to the line $2x + 4y - 7 = 0$.

6.49. Investigate the continuity and differentiability of the function

$$f(x) = \begin{cases} x^3 \sin\dfrac{1}{x},\ \text{if } x \neq 0; \\ 0,\ \text{if } x = 0. \end{cases}$$

6.50. Select the coefficients α and β so that the function

$$f(x) = \begin{cases} x^2,\ \text{if } x \le x_0; \\ \alpha x + \beta,\ \text{if } x > x_0 \end{cases}$$

was differentiable at a point x_0.

6.51. Find the derivative y' of the function $y = |x| + |x - 2|,\ x \neq 0, x \neq 2$.

6.52. Find the derivative $y'(x)$ of a function:

$$y(x) = \begin{cases} \dfrac{\sqrt{1 + 2x^3} - 1}{x^2},\ if\ \ x \neq 0; \\ 0,\ if\ \ x = 0. \end{cases}$$

Is the given function $y(x)$ continuous? Is its derivative $y'(x)$ a continuous function at the point $x = 0$?

Answers and instructions

6.1. The function is continuous, but not differentiable.

6.3. $-\dfrac{2\sin x \cdot \operatorname{arctg}(\cos x)}{1 + \cos^2 x} + \dfrac{\operatorname{ctg} x}{2\sqrt{(1 - \ln^2(\sin x))} \arcsin(\ln(\sin x))}$.

6.4. $\dfrac{2x^2}{1 - x^6}\sqrt[3]{\dfrac{1 + x^3}{1 - x^3}},\ |x| \neq 1$. **6.5.** $-\sin 2x \cdot \cos(\cos 2x)$.

6.6. $-\dfrac{1 + \cos^2 x}{2\sin^3 x},\ x \neq k\pi,\ k \in \mathbb{Z}$. **6.7.** $\dfrac{x^2}{(\cos x + x\sin x)^2}$.

6.8. $\dfrac{8}{3\sin^4 x\sqrt[3]{ctgx}}$, $x \neq k\pi$, $k \in \mathbb{Z}$. **6.9.** $\dfrac{e^x(\sin x - \cos x)}{2\sin^2 \dfrac{x}{2}}$, $x \neq 2k\pi$, $k \in \mathbb{Z}$.

6.10. $\dfrac{6}{x\ln x \ln(\ln^3 x)}$, $x > e$. **6.11.** $\dfrac{x}{x^4 - 1}$, $|x| > 1$.

6.12. $-ctg^3 x$, $0 < x - 2k\pi < \pi$, $k \in \mathbb{Z}$. **6.13.** $\sin x \cdot \ln tgx$, $0 < x - 2k\pi < \pi/2$, $k \in \mathbb{Z}$.

6.14. $\dfrac{\operatorname{sgn} x}{\sqrt{1 - x^2}}$, $0 < |x| < 1$. **6.15.** $\dfrac{1}{1 + x^2}$, $x \neq 1$.

6.16. $\dfrac{x\ln x}{(x^2 - 1)^{3/2}}$, $x > 1$. **6.17.** $\dfrac{x\arcsin x}{(1 - x^2)^{3/2}}$, $|x| < 1$. **6.18.** $\dfrac{e^x}{\sqrt{1 + e^{2x}}}$.

6.19. $th^3 x$. **6.20.** $\dfrac{1}{ch2x}$. **6.21.** 0.

6.22. $1 + \pi/4$. **6.23.** $-\sqrt[3]{\dfrac{y}{x}}$. **6.24.** $\dfrac{x + y}{x - y}$.

6.25. $\dfrac{y\sqrt{1 - (x + y)^2} - 1}{1 - x\sqrt{1 - (x + y)^2}}$. **6.26.** $\dfrac{y(x + y)\ln 10 - y^2}{y^2 + x(x + y)\ln 10}$.

6.28. The equation of the tangent to the curve: $y - 1 = \dfrac{4}{\ln 2}\left(x - \dfrac{\pi}{4}\right)$.

6.29. $x^{\frac{1}{x} - 2}(1 - \ln x)$, $x > 0$. **6.30.** $\left(\dfrac{x}{x+1}\right)^x \left(\ln \dfrac{x}{x+1} + \dfrac{1}{x+1}\right)$.

6.31. $(\cos 2x)^{tgx - 1}(\cos 2x \cdot \ln \cos 2x \cdot \sec^2 x - 4\sin^2 x)$.

6.32. $(\sin 8x)^{\cos 5x - 1}(8\cos 5x \cdot \cos 8x - 5\sin 8x \cdot \sin 5x \cdot \ln \sin 8x)$.

6.33. $\dfrac{2}{7}\sqrt[7]{\dfrac{e^{\sin 4x}(x^2 + 6x + 10)^3}{\operatorname{arctg}^2\left(\dfrac{1}{x}\right)\cdot \operatorname{sh}^6 2x}}\left(2\cos 4x + \dfrac{3x + 9}{x^2 + 6x + 10} + \dfrac{1}{x^2 + 1} - 6\operatorname{ch} 2x\right)$.

6.34. $\dfrac{54 - 36x + 4x^2 + 2x^3}{3x(1 - x)(9 - x^2)}$, $x \neq 0; 1; \pm 3$. **6.35.** -1. **6.36.** $\dfrac{b}{a}\operatorname{cth} t$, $|t| > 0$.

6.37. $\dfrac{3}{2}te^{-t^2}$. **6.38.** $\dfrac{1 - tgt}{1 + ctgt}$.

6.41. $\dfrac{\pi}{2}$; $\operatorname{arctg}\dfrac{3}{4} \approx 37^\circ$. **6.42.** $\alpha_1 = \operatorname{arctg}\left(\dfrac{e^3}{1 + 12 \cdot e^6}\right)$, $\alpha_2 = \operatorname{arctg}\left(\dfrac{\sqrt{e^3}}{1 + 6 \cdot e^3}\right)$.

6.43. $3x + 5y - 50 = 0$, $5x - 3y - 10{,}8 = 0$.

6.45. $3x - 2y = 0,\ 2x + 3y = 0$.

6.46. $3x - y - 4 = 0,\ x + 3y - 3 = 0$.

6.47. $3x + y - 3 = 0,\ 3x + y + 13 = 0$. **6.48.** $2x - y + 3 = 0$.

6.49. The function is continuous and differentiable.

6.50. $\alpha = 2x_0,\ \beta = -x_0^2.$

6.51. $y'(x) = \begin{cases} -2, if\ \ x < 0; \\ 0, if\ \ 0 < x < 2; \\ 2, if\ \ x > 2. \end{cases}$

6.52. $y'(x) = \begin{cases} \dfrac{2\sqrt{1+2x^3} - x^3 - 2}{x^3\sqrt{1+2x^3}}, if\ \ x \neq 0; \\ 1, if\ \ x = 0. \end{cases}$ Yes, both functions $y(x)$ and $y'(x)$ are ***continuous*** at the point $x = 0$.

PRACTICAL LESSON 7
The concept of differential.
Application of the differential in approximate calculations.
Derivatives and differentials of higher orders

We will briefly state the theoretical provisions related to the subject of this practical lesson.

7.1. The concept of differential

Definition 7.1. ***The differential*** dy *of a function* $y = f(x)$ *at a point* x *is the* ***main****,* ***linear relative*** *to* Δx *, part of* ***the increment of the function*** $y = f(x)$ *at this point:*

$$dy = f'(x)\Delta x. \tag{7.1}$$

If $y = x$, then $y' = x' = 1$, therefore $dy = dx = \Delta x$, that is, the differential dx of the independent variable x coincides with its increment Δx. Therefore, formula (7.1) can be written as follows:

$$dy = f'(x)dx. \tag{7.2}$$

Formula (7.2) allows you to consider the derivative $f'(x)$ as the ratio of the differential of the function to the differential of the independent variable (G. F. A. Leibnitz):

$$f'(x) = \frac{dy}{dx}.$$

7.2. The geometric content of the differential

The geometric content of the differential can be understood from Fig. 7.1. The following relations take place:

$$BD = \Delta y;\ BC = MB \cdot tg\alpha = f'(x)\Delta x = f'(x)dx = dy.$$

Thus, the differential of the function $y = f(x)$ at the given values x and Δx is equal to the increment of the ordinate of the tangent dy to the curve $y = f(x)$ at a point x on the interval Δx. At the same time, the increment of the function Δy is equal to the increment of the ordinate of the curve. For Δx specially chosen large scale.

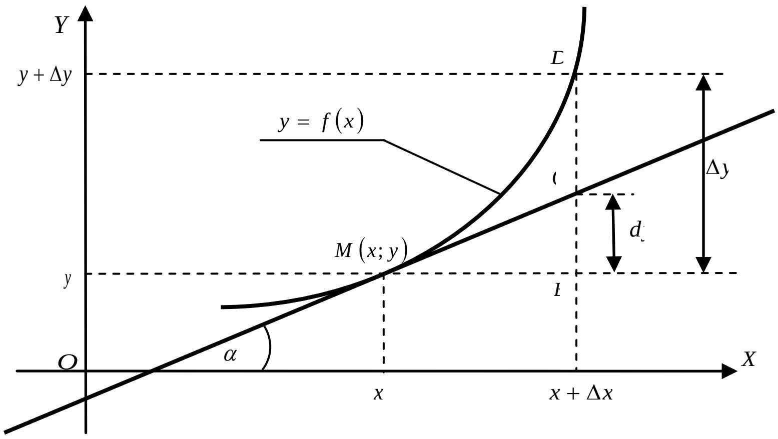

Fig. 7.1

Fig. 7.1

Therefore, replacing the increment of the function with its differential geometrically means replacing the increment of the function Δy (segment BD) on the interval $[x; \Delta x]$ with the increment of the ordinate of the tangent dy (segment BC) on the same interval. Naturally, such a replacement makes sense only for sufficiently small values Δx. The error we make when replacing Δy with dy, is equal to the length of the segment CD. In Fig. 7.1 is marked: BD – the increment of the function Δy, BC – the increment of the ordinate dy of the tangent drawn at the point $M\,(x; y)$.

7.3. The mechanical content of the differential

Let's explain the mechanical content of the differential. Let the material point move in a straight line according to the well-known law $S = S(t)$, where $S(t)-$ is a function differentiable over some interval, and t – is time. Then the differential $dS = \dot{S}(t)\Delta t$ of this function at fixed values of t and Δt – is the path that a material point would have traveled over time Δt if it had moved in a straight line and uniformly at a constant speed $V = \dot{S}(t)$. It is obvious that the actual path ΔS in the case of non-uniform movement, in contrast to the differential dS, is not a linear function of time Δt and therefore differs from the path dS. However, if the time interval Δt is small enough, the speed of movement does not have time to change significantly, and therefore the movement of the point during the time interval from t to $t + \Delta t$ is almost uniform.

7.4. The properties of the differential

Let $u(x)$ and $v(x)$ – be differentiable functions from x, $C = const$, then we have the following rules for finding differentials:

1) $dC = 0$;
2) $d(Cu) = Cdu$;
3) $d(u \pm v) = du \pm dv$;
4) $d(uv) = vdu + udv$;
5) $d\left(\dfrac{u}{v}\right) = \dfrac{vdu - udv}{v^2}$, $v \neq 0$.

The proof of the above formulas 1) – 5) is carried out similarly to the proof of the corresponding formulas for derivatives.

A particularly ***important property*** follows from the rule of differentiation of a ***composite function***. Let $y = f(x) = f(\varphi(t))$ is a composite function with an ***intermediate argument*** $x = \varphi(t)$ and a ***final argument*** t, and functions $f(x)$, $\varphi(t)$ differentiable at the points x and t. Then there is a derivative $y'_t = y'_x x'_t$, and therefore a differential

$$dy = y'_t dt = y'_x x'_t dt = y'_x dx. \tag{7.3}$$

Comparing formulas (7.2) and (7.3), we can conclude that the first differential of a function $y = f(x)$ is determined by the same formula regardless of whether a variable x is an independent variable or a function of another variable. This property of the differential is called ***invariance of the form of the first differential***.

Example 7.1. Find the tangent to the curve $\rho(\varphi) = 1 + \cos\varphi$ at the point $\varphi = \dfrac{\pi}{4}$.

The solution. Let us recall the formulas of the connection between the coordinates of the polar and Cartesian systems.

$$\begin{cases} y = \rho(\varphi)\sin\varphi; \\ x = \rho(\varphi)\cos\varphi. \end{cases}$$

We use the formula (7.3). First, we find the derivative $y'(x)$ as a ratio of differentials:

$$y'(x) = \frac{dy}{dx} = \frac{d[\rho(\varphi)\sin\varphi]}{d[\rho(\varphi)\cos\varphi]} = \frac{[\rho'(\varphi)\sin\varphi + \rho(\varphi)\cos\varphi]d\varphi}{[\rho'(\varphi)\cos\varphi - \rho(\varphi)\sin\varphi]d\varphi} =$$

$$= \frac{-\sin^2\varphi + \cos\varphi + \cos^2\varphi}{-\sin\varphi\cos\varphi - \sin\varphi - \sin\varphi\cos\varphi} = -\frac{\cos\varphi + \cos 2\varphi}{\sin\varphi + \sin 2\varphi}.$$

Let's calculate its value at a point $\varphi = \frac{\pi}{4}$:

$$y'(x)\Big|_{\varphi=\frac{\pi}{4}} = -\frac{\cos\varphi + \cos 2\varphi}{\sin\varphi + \sin 2\varphi}\Bigg|_{\varphi=\frac{\pi}{4}} = -\frac{\frac{\sqrt{2}}{2}+0}{\frac{\sqrt{2}}{2}+1} = -\frac{1}{\sqrt{2}+1} = 1-\sqrt{2}\,.$$

Now it is possible to write down the equation of the tangent, having previously found the coordinates of its point of contact: $x_0 = y_0 = \frac{1}{\sqrt{2}} + \frac{1}{2}$:

$$y - y_0 = y'(x_0)(x - x_0) \Rightarrow$$

$$y - \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right) = \left(1-\sqrt{2}\right)\left(x - \left(\frac{1}{\sqrt{2}} + \frac{1}{2}\right)\right) \Rightarrow\ y = \left(1-\sqrt{2}\right)x + 1 + \frac{1}{\sqrt{2}}\,.$$

7.5. Application of the first differential in approximate calculations

For small Δx increments Δy the function $y = f(x)$ at a point x can be approximately replaced by the differential dy at that point: $\Delta y \approx dy$. Substitute here the corresponding expressions for Δy and dy, as a result we get:

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x\,. \qquad (7.4)$$

We will show how formula (7.4) is used in approximate calculations.

Example 7.2. Prove that the formula is valid for small values Δx, $x > 0$ and for even numbers n : $\sqrt[n]{x + \Delta x} \approx \sqrt[n]{x} + \frac{\Delta x}{n\sqrt[n]{x^{n-1}}}$ ($n \geq 2$).

The solution. Consider the function $f(x) = \sqrt[n]{x}$, $x \in (0;+\infty)$ for even numbers n. Let's find the derivative $f'(x) = \frac{1}{n\sqrt[n]{x^{n-1}}}$ and the desired equality follows from formula (7.4). In particular, if $x = 1$ then $\sqrt[n]{1+\Delta x} \approx 1 + \frac{\Delta x}{n}$.

Based on the found formula, we will approximately calculate the given expressions:

a) $\sqrt{1,06} = \sqrt{1 + 0,06} \approx 1 + \frac{0,06}{2} = 1,03$;

b) $\sqrt{142} = \sqrt{144 - 2} = \sqrt{144\left(1 - \frac{2}{144}\right)} = 12\sqrt{1 - \frac{2}{144}} \approx 12 \cdot \left(1 - \frac{1}{144}\right) = 11,917$

Example 7.3. Calculate approximately $\arcsin(0,48)$.

The solution*. Importantly!** In such examples, angles should be considered in ***radians.

Let $f(x) = \arcsin x$, then by formula (7.4) we have

$$\arcsin(x + \Delta x) \approx \arcsin x + (\arcsin x)' \Delta x;$$

$$\arcsin(x + \Delta x) \approx \arcsin x + \frac{\Delta x}{\sqrt{1 - x^2}}.$$

Because $x = 0,5,\ \Delta x = -0,02$, then

$$\arcsin 0,48 \approx \arcsin 0,5 - \frac{0,02}{\sqrt{1 - 0,25}} = \frac{\pi}{6} - \frac{0,04}{\sqrt{3}} = 0,524 - 0,023 \approx 0,501.$$

Example 7.4. Calculate approximately $\operatorname{arctg}(1,05)$.

***The solution*.** Let $f(x) = \operatorname{arctg} x$, then by formula (7.4) we have:

$$\operatorname{arctg}(x + \Delta x) \approx \operatorname{arctg} x + (\operatorname{arctg} x)' \Delta x;$$

$$\operatorname{arctg}(x + \Delta x) \approx \operatorname{arctg} x + \frac{\Delta x}{1 + x^2}.$$

In our case, we have: $x = 1,\ \Delta x = 0,05$, $\operatorname{arctg} 1 = \pi/4$. Let's substitute these values in the above formula. As a result, we get:

$$\operatorname{arctg}(1,05) = \frac{\pi}{4} + \frac{0,05}{2} \approx 0,8104.$$

7.6. Derivatives and differentials of higher orders

Definition 7.2. *The* n *– order derivative of a function* $y = f(x)$ *is called the first derivative, if it exists, of the* $(n-1)$ *– order derivative:*

$$y^{(n)} = \left(y^{n-1}\right)', \text{ або } f^{(n)}(x) = \left(f^{(n-1)}(x)\right)', \text{ або } \frac{d^n y}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1} y}{dx^{n-1}}\right). \qquad (7.5)$$

Example 7.5. Find the fourth-order derivative for the function $y = x^5 - 6x^3 + 3x^2 - 1$.

***The solution*.** According to formula (7.5), we have:

$$y' = 5x^4 - 18x^2 + 6x,\ y'' = 20x^3 - 36x + 6,\ y''' = 60x^2 - 36,\ y^{(4)} = 120x.$$

Example 7.6. Find the n – order derivative of the function $y = e^{kx}$.

***The solution*.** $y' = ke^{kx},\ y'' = k^2 e^{kx},\ y''' = k^3 e^{kx}, \cdots, y^{(n)} = k^n e^{kx}$.

Example 7.7. Find the 11-order derivative of the function $y = \mathrm{arctg}\, x$.

The solution. First, let's make the following transformations (here *i* is an imaginary unit):

$$y' = (\mathrm{arctg}\, x)' = \frac{1}{1+x^2} = \left| x = iz \right| = \frac{1}{1-z^2} = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right) = \frac{1}{2}\left((1-z)^{-1} + (1+z)^{-1}\right).$$

Now we can simply find the tenth-order derivatives for each term in parentheses:

$$[(1-z)^{-1}]^{(10)} = \frac{10!}{(1-z)^{11}},$$

$$[(1+z)^{-1}]^{(10)} = \frac{10!}{(1+z)^{11}}.$$

Let's return to the old variable and write down the differentiation result:

$$y^{(11)} = \frac{10!}{2}\left(\frac{1}{\left(1-\frac{x}{i}\right)^{11}} + \frac{1}{\left(1+\frac{x}{i}\right)^{11}}\right) = -\frac{10!}{2}\frac{[(1+ix)^{11} + (1-ix)^{11}]}{(1+x^2)^{11}}.$$

After opening the brackets, we will get the final result:

$$y^{(11)} = 10! \cdot \frac{11x^{10} - 165x^8 + 462x^6 - 330x^4 + 55x^2 - 1}{(1+x^2)^{11}}.$$

7.7. The Leibnitz's formula

The Leibnitz's formula is used to find derivatives of higher orders from the product of two differentiable functions.

Theorem. *Let the functions $u(x)$ and $v(x)$ have derivatives up to and including the n – th order at the point x, then the following formula holds:*

$$\left(u \cdot v\right)^{(n)} = \sum_{k=0}^{n} C_n^k u^{(k)} v^{(n-k)}, \qquad (7.6)$$

where $C_n^k = \dfrac{n!}{k!(n-k)!} = \dfrac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)}{k!}$ *– binomial coefficients. Formula (7.6) is called **the Leibnitz's formula**.*

Example 7.8. Find the third derivative of the product $y(x) = x^2 e^x$.

The solution. According to formula (7.6), we have:

$$\left(x^2e^x\right)''' = \sum_{k=0}^{3} C_3^k (x^2)^{(k)}(e^x)^{(3-k)} = x^2e^x + C_3^1\cdot 2x\cdot e^x + C_3^2\cdot 2\cdot e^x + C_3^3\cdot 0\cdot e^x = e^x(x^2+6x+6).$$

So, we have:

$$y'''(x) = \left(x^2e^x\right)''' = e^x(x^2+6x+6).$$

Example 7.9. Find the value of the third derivative of the function $y(x) = (x^3+2x^2+x+1)\sin 2x$ at a point $x = 0$.

The solution. According to formula (7.6), we have:

$$\left((x^3+2x^2+x+1)\sin 2x\right)''' = \sum_{k=0}^{3} C_3^k (x^3+2x^2+x+1)^{(k)}(\sin 2x)^{(3-k)} =$$

$$= (x^3+2x^2+x+1)(-8\cos 2x) + C_3^1\cdot(3x^2+4x+1)\cdot(-4\sin 2x) + C_3^2\cdot(6x+4)\cdot 2\cos 2x +$$

$$+C_3^3\cdot 6\cdot\sin 2x = (-8x^3-16x^2+28x+16)\cos 2x - 6(6x^2+8x+1)\sin 2x.$$

So, we have:

$$y'''(0) = \left[(-8x^3-16x^2+28x+16)\cos 2x - 6(6x^2+8x+1)\sin 2x\right]\Big|_{x=0} = 16.$$

7.8. Derivatives of higher orders of an implicitly given function

Example 7.10. Find $y''(x)$, if $x^2+\sin y = y$.

The solution. The procedure for finding derivatives of higher orders for unspecified functions is similar to the same procedure for finding the derivative of the first order. We differentiate the given equality with respect to the variable and find the first derivative $y'(x)$:

$$2x + \cos y\cdot y' = y' \Rightarrow y' = \frac{2x}{1-\cos y}.$$

We differentiate by variable x the resulting expression for $y'(x)$ and find $y''(x)$:

$$y''(x) = 2\cdot\frac{1-\cos y - x\sin y\cdot y'}{(1-\cos y)^2} = 2\cdot\frac{1-\cos y - x\sin y\cdot\dfrac{2x}{1-\cos y}}{(1-\cos y)^2} =$$

$$= 2\cdot\frac{(1-\cos y)^2 - 2x^2\sin y}{(1-\cos y)^3}.$$

7.9. Derivatives of higher orders of a parametrically specified function

Example 7.11. Find $y^{(n)}(x)$, if $x(t)=\ln t,\ y(t)=t^2$.

The solution. We will successively find the first, second, etc., $y^{(n)}(x)$ derivatives

$$y'_x=\frac{dy}{dx}=\frac{y'(t)}{x'(t)}=\frac{2t}{1/t}=2t^2;\quad \frac{d^2y}{dx^2}=\frac{\left(y'_x\right)'_t}{x'(t)}=\frac{4t}{1/t}=4t^2;$$

$$\frac{d^3y}{dx^3}=\frac{\left(y''_x\right)'_t}{x'(t)}=\frac{8t}{1/t}=8t^2;\ \ldots;\frac{d^{(n)}y}{dx^{(n)}}=\frac{2^n t}{1/t}=2^n t^2.$$

Example 7.12. Find $y'''_{x^3}(x)$, if $x=\cos\left(\frac{t}{2}\right),\quad y=t-\sin t$.

The solution. Let's find the first derivative by variable x:

$$y'_x=\frac{(t-\sin t)'_t}{\left(\cos\left(\frac{t}{2}\right)\right)'_t}=\frac{1-\cos t}{-\frac{1}{2}\sin\left(\frac{t}{2}\right)}=-4\sin\left(\frac{t}{2}\right).$$

Now we will find the second derivative of the variable x:

$$y''_{x^2}=\frac{\left(-4\sin\left(\frac{t}{2}\right)\right)'_e}{\left(\cos\left(\frac{t}{2}\right)\right)'_t}=\frac{-2\cos\left(\frac{t}{2}\right)}{-\frac{1}{2}\sin\left(\frac{t}{2}\right)}=4\,\mathrm{ctg}\left(\frac{t}{2}\right).$$

Let's find the third derivative by variable x:

$$y'''_{x^3}=\frac{\left(4\,\mathrm{ctg}\left(\frac{t}{2}\right)\right)'_e}{\left(\cos\left(\frac{t}{2}\right)\right)'_t}=\frac{\dfrac{-2}{\sin^2\left(\frac{t}{2}\right)}}{-\frac{1}{2}\sin\left(\frac{t}{2}\right)}=\frac{4}{\sin^3\left(\frac{t}{2}\right)}.$$

7.10. Differentials of higher orders

Definition 7.3. ***The second differential*** d^2y*, or second-order differential, is the differential from the first differential:* $d^2y = d(dy)$.

Since dx *does not depend on* x*, then when differentiating the first differential, the factor* dx *can be taken beyond the sign of the derivative, therefore*

$$d^2y = d(dy) = d\left(f'(x)dx\right) = \left(f'(x)dx\right)'_x dx = f''(x)(dx)^2. \qquad (7.7)$$

Definition 7.4. *A differential of the* n *– th order* d^ny *is called a differential from a differential of the* $(n-1)$ *– th order:*

$$d^ny = d\left(d^{(n-1)}y\right) = f^{(n)}(x)dx^n. \qquad (7.8)$$

Remark. Formulas (7.7) – (7.8) are valid only for the case when x – ***the independent variable***. Indeed, let the composite function $y = f(x)$, $x = x(t)$. It is known that the ***first differential has an invariant form***, that is, the equality $dy = f'(x)dx$ holds both for the case when x is a function of t, and for the condition that x is an independent variable.

It turns out that ***differentials of higher orders do not have an invariant property***. Let's show it on the example of a second-order differential. Using the rule of product differentiation, we have:

$$d^2y = d\left(f'(x)dx\right) = d\left(f'(x)\right)dx + f'(x)d(dx) = f''(x)dx^2 + f'(x)d^2x =$$
$$= f''(x)dx^2 + f'(x)x''(t)dt^2.$$

So,

$$d^2y = f''(x)dx^2 + f'(x)x''(t)dt^2. \qquad (7.9)$$

Example 7.13. Find d^3y, if $y(x) = \cos 3x$.

The solution. Let's use formula (7.8).

Since $y' = -3\sin 3x,\ y'' = -9\cos 3x,\ y''' = 27\sin 3x$, then $d^3y = 27\sin 3x \cdot dx^3$.

Example 7.14. Find d^2y if $y(x) = x^3;\ x(t) = t^2 + 1$.

The solution. Let's use formula (7.8):

$f(x) = x^3;\ f'(x) = 3x^2;\ f''(x) = 6x;\ x'(t) = 2t;\ x''(t) = 2$,

$dx = x'(t)dt = 2tdt$,

then

$$d^2y = 6xdx^2 + 3x^2 \cdot 2 \cdot dt^2 = 6\left(t^2+1\right)(2tdt)^2 + 6\left(t^2+1\right)^2(dt)^2 = \left(30t^4 + 36t^2 + 6\right)dt^2.$$

We will get the same result if we immediately exclude the dependent variable x and use formula (7.7):

$y(x) = x^3;\ x(t) = t^2 + 1$, тому

$y(t) = \left(t^2 + 1\right)^3;\ y_t' = 3\left(t^2 + 1\right)^2 2t = 6\left(t^5 + 2t^3 + t\right);$

$y_{tt}'' = 6(5t^4 + 6t^2 + 1);\quad d^2y = (30t^4 + 36t^2 + 6)dt^2.$

Tasks for classroom and independent work

7.1. Find the differential and gain of a function $y = x^3$ at the point $x = 1$ at $\Delta x = 0{,}1$. What will be the relative error in case of replacing the gain of a function with its differential?

7.2. Substantiate the general formula for the approximate calculation of function values using a differential and prove approximate formulas:

a) $\sqrt[n]{x + \Delta x} \approx \sqrt[n]{x} + \dfrac{\Delta x}{n\sqrt[n]{x^{n-1}}},\ n \geq 2$; b) $\ln(x + \Delta x) \approx \ln x + \dfrac{\Delta x}{x}$;

c) $\sin(x + \Delta x) \approx \sin x + \Delta x \cdot \cos x$; d) $\cos(x + \Delta x) \approx \cos x - \Delta x \cdot \sin x$.

I. Find the differential of a function $y = f(x)$:

7.3. $y = \sin^3 \dfrac{6x}{x+1}$. **7.4.** $y = 5^{\sin x \cos 2x}$.

II. Find the differential of a function $y = y(x)$, defined implicity:

7.5. $e^{y^2} = x^2 - y$. **7.6.** $2xy = \ln(x^2 - 2y)$.

III. Approximately calculate the value of the specified functions (with accuracy $\varepsilon = 10^{-3}$):

7.7. a) $\cos 151^0$; b) $\operatorname{arctg} 0{,}98$. **7.8.** a) $\sin 29^0$; b) $\ln(e - 0{,}02)$.

IV. Find the second derivative of the function $y = f(x)$:

7.9. $y = (x^2 + 1)\operatorname{arctg} x$. **7.10.** $y = x^2 \ln x$.

V. Find $y''(0)$, if:

7.11. $y = e^{3x} \sin 2x$. **7.12.** $y = e^{2x} \ln(x + 1)$.

VI. Find the second derivative $\frac{d^2y}{dx^2}$ of the function $y = f(x)$, defined parametrically:

7.13. $x = 3t - t^2,\ y = 4t - t^3$.

7.14. $x = a\cos t,\ y = a\sin t$.

VII. Find $y''(x)$, if the function $y = y(x)$ is implicitly defined:

7.15. $e^{x-y} = x + y$.

7.16. $\sqrt{x^2 + y^2} = ae^{arctg\frac{y}{x}},\ a > 0$.

VIII. Find the derivative of the specified order of the function $y = f(x)$:

7.17. $y = \frac{x^2}{1-x},\ y^{(8)} = ?$

7.18. $y = x\ln x,\ y^{(6)} = ?$

7.19. $y = x\,\text{sh}\,x,\ y^{(100)} = ?$

7.20. $y = e^x\cos x,\ y^{(4)} = ?$

IX. Find the derivative of the $n-$ order from the function $y = f(x)$:

7.21. $y = \frac{1}{x(1-x)}$.

7.22. $y = \frac{x}{\sqrt[3]{x+1}}$.

7.23. $y = \cos^2 x$.

7.24. $y = e^x \sin x$.

X. Find $f^{(n)}(0)$, if:

7.25. $f(x) = \frac{1}{(1-2x)(1+x)}$.

7.26. $f(x) = x^2 e^{ax}$.

XI. Find differential of indicated order from the function $y = f(x)$:

7.27. $y = \frac{1}{\sqrt{x}},\ d^3y = ?$

7.28. $y = e^x\ln x,\ d^4y = ?$

XII. Show that the function $y = f(x)$ satisfied the given differential equation:

7.29. $y = e^x + 2e^{2x}$, $y''' - 6y'' + 11y' - 6y = 0$.

7.30. $y = x\ln^2 x$, $y'' - \frac{y'}{x} + \frac{y}{x} = \frac{2}{x}$.

7.31. Prove by induction:

$$y^{(n)} = \left(\frac{ax+b}{cx+d}\right)^{(n)} = (ad - cb)(-1)^{n+1} n!(cx+d)^{-(n+1)} c^{n-1}.$$

7.32. Prove that the function $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0 \end{cases}$ is infinitely differentiated at the point $x = 0$.

7.33. Prove that $d^n(\sin^2 x) = 2^{n-1}\sin\left(2x + \frac{\pi(n-1)}{2}\right)dx^n$.

7.34. Find $d^n y$, if $y(x) = \frac{\ln x}{x}$.

XIII. By mathematical induction to prove the formulas:

7.35. $(a^x)^{(n)} = a^x \ln^n a,\ a > 0$. **7.36.** $(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right)$.

7.37. $(x^m)^{(n)} = m(m-1)(m-2)\cdots(m-n+1)x^{m-n}$.

7.38. $(\ln x)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$.

XIV. Calculate derivatives

7.39. $\frac{d}{d(x^2)}\left(\frac{\sin x}{x}\right)$. **7.40.** $\frac{d(\operatorname{tg} x)}{d(\operatorname{ctg} x)}$. **7.41.** $\left.\frac{d(x\cdot\sin x)}{d(x\cdot\cos x)}\right|_{x=\frac{\pi}{4}}$. **7.42.** $\left.\frac{d(\operatorname{sh}(\operatorname{tg} x))}{d(\sin x)}\right|_{x=\frac{\pi}{4}}$.

7.43. Using Leibnitz's formula, find the tenth-order derivative of a function $y = e^{3x}\cos 2x$. **Hint**. Present the trigonometric function according to Euler's formula: $\cos 2x = \frac{e^{2xi} + e^{-2xi}}{2}$.

7.44. Prove that the second derivative of the function $y(x)$ given parametrically by the equations $y = y(t), x = x(t)$ is found by the formula:

$$\frac{d^2y}{dx^2} = \frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^3}.$$

Answers and instructions

7.3. $dy = \frac{9}{(x+1)^2}\sin\frac{12x}{x+1}\sin\frac{6x}{x+1}dx$.

7.4. $dy = \frac{1}{2}5^{\sin x\cos 2x}(3\cos 3x - \cos x)\ln 5dx$.

7.5. $dy = \frac{2xdx}{2ye^{y^2}+1}$. **7.6.** $dy = \frac{x - x^2y + 2y^2}{x^3 - 2xy + 1}dx$.

7.7. $а)\,-0{,}8747$; $б)\ 0{,}7754$. **7.8.** $а)\ 0{,}4849$. **7.9.** $\dfrac{2x}{1+x^2}+2\,\mathrm{arctg}\,x$.

7.10. $y''=2\ln x+3$. **7.11.** 12. **7.12.** 3.

7.13. $y''=\dfrac{6t^2-18t+8}{(3-2t)^3}$. **7.14.** $y''=-\dfrac{1}{a\sin^3 t}$, $t\neq k\pi$, $k\in\mathbb{Z}$.

7.15. $\dfrac{4e^{x-y}}{(1+e^{x-y})^3}$. **7.16.** $y'=\dfrac{x+y}{x-y}$, $y''=\dfrac{2(x^2+y^2)}{(x-y)^3}$.

7.17. $y^{(8)}=\dfrac{8!}{(1-x)^9}$, $x\neq 1$. **7.18.** $y^{(6)}=\dfrac{4!}{x^5}$.

7.19. $y^{(100)}=x\cdot shx+100chx$. **7.20.** $y^{(4)}=-4e^x\cos x$.

7.21. $n!\left(\dfrac{(-1)^n}{x^{n+1}}+\dfrac{1}{(1-x)^{n+1}}\right)$.

7.22. $(-1)^{n+1}\dfrac{1\cdot 4\cdots(3n-5)(3n+2x)}{3^n(1+x)^{n+1/3}}$, $n\geq 2$, $x\neq -1$.

7.23. $2^{n-1}\cos(2x+n\pi/2)$. **7.24.** $e^x 2^{n/2}\sin\left(x+\dfrac{n\pi}{4}\right)$.

7.25. $\dfrac{n!}{3}\left(2^{n+1}+(-1)^n\right)$. **7.26.** $n(n-1)a^{n-2}$.

7.27. $-\dfrac{15}{8x^3\sqrt{x}}dx^3$, $x>0$. **7.28.** $e^x\left(\ln x+\dfrac{4}{x}-\dfrac{6}{x^2}+\dfrac{8}{x^3}-\dfrac{6}{x^4}\right)dx^4$.

7.34. $(-1)^n\dfrac{n!}{x^{n+1}}\left(\ln x-\sum\limits_{k=1}^{n}\dfrac{1}{k}\right)dx^n$, $x>0$.

7.39. $\dfrac{x\cdot\cos x-\sin x}{2x^3}$. **7.40.** $-\mathrm{tg}^2 x$, $x\neq\dfrac{\pi}{2}+\pi n$.

7.41. $\dfrac{\pi+4}{4-\pi}$. **7.42.** $\left.\dfrac{\mathrm{ch}\left(\mathrm{tg}\,x\right)}{\cos^3 x}\right|_{x=\frac{\pi}{4}}=2\sqrt{2}\cdot\mathrm{ch}\left(1\right)$.

PRACTICAL LESSON 8
The Lhospital–Bernoulli rule. Formulas of Taylor and Maclaurin

We will present important theoretical provisions related to the subject of this practical lesson.

8.1. The main theorems justifying the application of the Lhospital–Bernoulli rule

Theorem 8.1. *Let the functions* $f(x), \varphi(x)$ *be defined and differentiable around the point* x_0*, with the possible exception of the point* x_0 *itself, and*

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \varphi(x) = 0,$$

and in the specified neighborhood $\varphi'(x) \neq 0$. *Then if there is a limit of the ratio of derivatives* $\lim\limits_{x\to x_0}\frac{f'(x)}{\varphi'(x)}$*, then there is also a limit of the ratio of functions* $\lim\limits_{x\to x_0}\frac{f(x)}{\varphi(x)}$ *and these limits are equal to each other:*

$$\lim_{x\to x_0}\frac{f(x)}{\varphi(x)} = \lim_{x\to x_0}\frac{f'(x)}{\varphi'(x)}. \tag{8.1}$$

Remarks 8.1. The theorem is also valid in the case when $x_0 \to \infty$. Indeed, putting $x = z^{-1}$, we have

$$\lim_{x\to\infty}\frac{f(x)}{\varphi(x)} = \lim_{z\to 0}\frac{f\left(z^{-1}\right)}{\varphi\left(z^{-1}\right)} = \lim_{z\to 0}\frac{\left(f\left(z^{-1}\right)\right)'}{\left(\varphi\left(z^{-1}\right)\right)'} = \lim_{z\to 0}\frac{f'\left(z^{-1}\right)\left(-z^{-2}\right)}{\varphi'\left(z^{-1}\right)\left(-z^{-2}\right)} = \lim_{x\to\infty}\frac{f'(x)}{\varphi'(x)}. \tag{8.2}$$

Remark 8.2. If the derivatives $f'(x), \varphi'(x)$ satisfy the same conditions as the functions $f(x), \varphi(x)$, then Theorem 8.1 ***can be applied again***. At the same time, we get:

$$\lim_{x\to x_0}\frac{f(x)}{\varphi(x)} = \lim_{x\to x_0}\frac{f'(x)}{\varphi'(x)} = \lim_{x\to x_0}\frac{f''(x)}{\varphi''(x)}.$$

In general, Theorem 8.1 can be applied until we arrive at the relation of derivatives $\dfrac{f^{(n)}(x)}{\varphi^{(n)}(x)}$, which has a certain limit at $x \to x_0$, that is, until the uncertainty $\{0/0\}$ disappears. The ratio of functions will have the same limit:

$$\lim_{x\to x_0} \frac{f(x)}{\varphi(x)} = \lim_{x\to x_0} \frac{f^{(n)}(x)}{\varphi^{(n)}(x)}. \quad (8.3)$$

It is obvious that Theorem 8.1 makes it possible to reveal the uncertainty of the form $\{0/0\}$. We present the Theorem concerning the disclosure of uncertainty of the form $\{\infty/\infty\}$.

Theorem 8.2. *Let the functions* $f(x), \varphi(x)$ *be defined and differentiable in the neighborhood of a point* x_0 *and in this neighborhood the boundaries are equal*

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} \varphi(x) = \infty, \quad \varphi'(x) \neq 0.$$

Then if there is a limit of the ratio of derivatives $\lim\limits_{x\to x_0} \dfrac{f'(x)}{\varphi'(x)}$, then there is also a limit of the ratio of functions $\lim\limits_{x\to x_0} \dfrac{f(x)}{\varphi(x)}$ and these limits are equal to each other:

$$\lim_{x\to x_0} \frac{f(x)}{\varphi(x)} = \lim_{x\to x_0} \frac{f'(x)}{\varphi'(x)}. \quad (8.4)$$

An effective way of calculating the limits (8.1) – (8.4), substantiated by ***Theorems 8.1*** and ***Theorems 8.2***, is called ***Lhospital's rule*** after the mathematician (***G. F. A. Lhospital***) who published it. But this rule was discovered independently of **Lhospital** by Joh. Bernoulli, so the specified rule is also called ***the Lhospital–Bernoulli rule***.

However, one should not think that this rule allows calculating any limits. There are ***limits*** that cannot be found using this rule.

Task for IWS: come up with an example of a limit that cannot be determined by Lhospital's rule.

Remark 8.3. Lhospital's rule according to formulas (8.1) – (8.4) is directly applied only to reveal uncertainties of the form $\{0/0\}$, $\{\infty/\infty\}$, which are called ***basic*** (***classical***). However, there are also such uncertainties as, for example $\{0\cdot\infty\}$, $\{\infty-\infty\}$, $\{\infty^0\}$, $\{1^\infty\}$, $\{0^0\}$. Before applying the Lhospital–Bernoulli rule to them, they need to be reduced to the main ones. We will show how these uncertainties are reduced to the main ones.

1. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} \varphi(x) = \infty$, then the uncertainty of the species $\{0\cdot\infty\}$ can be reduced to the main ones as follows:

$$\lim_{x\to x_0}\left[f(x)\varphi(x)\right] = \lim_{x\to x_0}\frac{f(x)}{1/\varphi(x)} = \left\{\frac{0}{0}\right\},$$

$$\text{або } \lim_{x\to x_0}\left[f(x)\varphi(x)\right] = \lim_{x\to x_0}\frac{\varphi(x)}{1/f(x)} = \left\{\frac{\infty}{\infty}\right\}.$$

2. If $\lim_{x\to x_0} f(x) = \infty$ and $\lim_{x\to x_0} \varphi(x) = \infty$, then the uncertainty of the species $\{\infty-\infty\}$ reduces to the uncertainty of the species $\{0/0\}$ as follows:

$$f(x) - \varphi(x) = \frac{1/\varphi(x) - 1/f(x)}{1/\varphi(x)\cdot 1/f(x)}.$$

3. If $\lim_{x\to x_0} f(x) = 0$ and $\lim_{x\to x_0} \varphi(x) = 0$, then by means of transformation

$$\lim_{x\to x_0}\left[f(x)\right]^{\varphi(x)} = e^{\lim\limits_{x\to x_0}\left[\varphi(x)\cdot\ln f(x)\right]}$$

the uncertainty of the type $\{0^0\}$ is reduced to the uncertainty $\{0\cdot\infty\}$ discussed above. Uncertainties of the type $\{\infty^0\}$ and $\{1^\infty\}$ are revealed similarly.

Therefore, in order to reveal the uncertainties $\{0\cdot\infty\}, \{\infty-\infty\}, \{\infty^0\}$, $\{1^\infty\}$, $\{0^0\}$, they must first be reduced to ***the main ones*** and only then apply Lhospital's rule.

Remark 8.4. When finding, for example, limits of this type

$$\lim_{x\to x_0}\frac{\psi(x)f(x)}{\varphi(x)} \text{ або } \lim_{x\to x_0}\frac{f(x)}{\xi(x)\varphi(x)},$$

where $\lim_{x\to x_0}\psi(x) = A\ (A\neq 0;\infty)$, $\lim_{x\to x_0}\frac{1}{\xi(x)} = B\ (B\neq 0;\infty)$, $\lim_{x\to x_0} f(x) = \lim_{x\to x_0}\varphi(x) = 0$, the Lhospital's rule should be applied as follows:

$$\lim_{x\to x_0}\frac{\psi(x)f(x)}{\varphi(x)} = \lim_{x\to x_0}\psi(x)\cdot\lim_{x\to x_0}\frac{f'(x)}{\varphi'(x)} = A\cdot\lim_{x\to x_0}\frac{f'(x)}{\varphi'(x)}.$$

$$\lim_{x\to x_0}\frac{f(x)}{\xi(x)\varphi(x)} = \lim_{x\to x_0}\frac{1}{\xi(x)}\cdot\lim_{x\to x_0}\frac{f'(x)}{\varphi'(x)} = B\cdot\lim_{x\to x_0}\frac{f'(x)}{\varphi'(x)}.$$

This remark is also valid in the case of repeated application of Lhospital's rule.

8.2. The technique of applying the Lhospital–Bernoulli rule

Examples. Find limits:

1. $\lim\limits_{x\to 0}\dfrac{e^{7x}-e^{3x}}{x}$.

2. $\lim\limits_{x\to 3}(3-x)\,\mathrm{tg}\left(\dfrac{\pi x}{6}\right)$.

3. $\lim\limits_{x\to 0}\left(\dfrac{2\sin x}{\mathrm{arctg}(x^3)}-\dfrac{\sin 2x}{\arcsin^3(x)}\right)$.

4. $\lim\limits_{x\to +0}(\mathrm{ctg}\,x)^{\frac{1}{\ln x}}$.

5. $\lim\limits_{x\to 1}\left(\mathrm{tg}\left(\dfrac{\pi x}{4}\right)\right)^{\mathrm{tg}\left(\frac{\pi x}{2}\right)}$.

6. $\lim\limits_{x\to \frac{\pi}{2}}(\pi-2x)^{\cos x}$.

7. $\lim\limits_{x\to \infty} x^n e^{-2x},\ n\in N$.

The solution.

1. We have a basic uncertainty $\{0/0\}$, therefore:

$$\lim_{x\to 0}\frac{e^{7x}-e^{3x}}{x}=\lim_{x\to 0}\frac{7e^{7x}-3e^{3x}}{1}=4.$$

2. Here we have uncertainty $\{0\cdot\infty\}$. We will reduce it to the basic uncertainty of the form $\{0/0\}$, after which we will apply Lhospital's rule:

$$\lim_{x\to 3}(3-x)\,\mathrm{tg}\left(\frac{\pi x}{6}\right)=\lim_{x\to 3}\frac{(3-x)}{\mathrm{ctg}\left(\frac{\pi x}{6}\right)}=\left\{\frac{0}{0}\right\}=\lim_{x\to 3}\frac{-6}{-\pi\sin^{-2}\left(\frac{\pi x}{6}\right)}=\frac{6}{\pi}.$$

3. In this example, we have uncertainty $\{\infty-\infty\}$. Let's reduce it to the basic uncertainty of the species $\{0/0\}$ and use Lhospital's rule three times:

$$\lim_{x\to 0}\left(\frac{2\sin x}{\mathrm{arctg}(x^3)}-\frac{\sin 2x}{\arcsin^3(x)}\right)=\left|\begin{array}{l}\mathit{if}\ \ x\to 0\ \mathit{then}\\ \mathrm{arctg}(x^3)\sim x^3;\\ \arcsin^3(x)\sim x^3\end{array}\right|=\lim_{x\to 0}\left(\frac{2\sin x-\sin 2x}{x^3}\right)=\left\{\frac{0}{0}\right\}=$$

$$= \lim_{x\to 0}\frac{2\cos x-2\cos 2x}{3x^2}=\left\{\frac{0}{0}\right\}=\lim_{x\to 0}\frac{-2\sin x+4\sin 2x}{6x}=\left\{\frac{0}{0}\right\}=$$

$$=\lim_{x\to 0}\frac{-2\cos x+8\cos 2x}{6}=1.$$

4. Here we have uncertainty of the form $\{\ \infty^0\ \}$. With the help of known transformations, we will reduce the given uncertainty to the basic form $\{\infty/\infty\}$, after which we will apply Lhospital's rule:

$$\lim_{x\to +0}(\operatorname{ctg} x)^{\frac{1}{\ln x}}=\left\{\infty^0\right\}=e^{\lim\limits_{x\to +0}\left(\frac{\ln \operatorname{ctg} x}{\ln x}\right)}=\left\{\frac{\infty}{\infty}\right\}=e^{\lim\limits_{x\to +0}\left(\frac{\operatorname{tg} x\cdot\left(-\frac{1}{\sin^2 x}\right)}{\frac{1}{x}}\right)}=e^{-\lim\limits_{x\to +0}\left(\frac{x}{\sin x\cdot\cos x}\right)}=e^{-1}.$$

5. In this example, we have an uncertainty of the form $\{\ 1^\infty\ \}$. We will use the previous transformation to move to the main uncertainty $\{0/0\}$. After that, we will apply Lhospital's rule:

$$\lim_{x\to 1}\left(\operatorname{tg}\left(\frac{\pi x}{4}\right)\right)^{\operatorname{tg}\left(\frac{\pi x}{2}\right)}=e^{\lim\limits_{x\to 1}\operatorname{tg}\left(\frac{\pi x}{2}\right)\ln\left(\operatorname{tg}\left(\frac{\pi x}{4}\right)\right)}=\left\{\infty\cdot 0\right\}=e^{\lim\limits_{x\to 1}\frac{\ln\left(\operatorname{tg}\left(\frac{\pi x}{4}\right)\right)}{\operatorname{ctg}\left(\frac{\pi x}{2}\right)}}=\left\{\frac{0}{0}\right\}=$$

$$=e^{\lim\limits_{x\to 1}\frac{\operatorname{ctg}\left(\frac{\pi x}{4}\right)\cdot\cos^{-2}\left(\frac{\pi x}{4}\right)\cdot\frac{\pi}{4}}{-\sin^{-2}\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}}}=e^{-\lim\limits_{x\to 1}\frac{\sin^2\left(\frac{\pi x}{2}\right)}{2\cos\left(\frac{\pi x}{4}\right)\cdot\sin\left(\frac{\pi x}{4}\right)}}=e^{-\lim\limits_{x\to 1}\frac{\sin^2\left(\frac{\pi x}{2}\right)}{\sin\left(\frac{\pi x}{2}\right)}}=e^{-\lim\limits_{x\to 1}\sin\left(\frac{\pi x}{2}\right)}=e^{-1}.$$

6. Here we have uncertainty of the form $\{\ 0^0\ \}$, then

$$\lim_{x\to\frac{\pi}{2}}(\pi-2x)^{\cos x}=e^{\lim\limits_{x\to\frac{\pi}{2}}\cos x\cdot\ln(\pi-2x)}=\left\{0\cdot\infty\right\}=e^{\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{\ln(\pi-2x)}{\cos^{-1}x}\right]}=\left\{\frac{\infty}{\infty}\right\}=$$

$$=e^{\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{\frac{-2}{\pi-2x}}{\cos^{-2}x\cdot\sin x}\right]}=e^{-2\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{\cos^2 x}{\sin x\cdot(\pi-2x)}\right]}=e^{-2\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{1}{\sin x}\right]\cdot\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{\cos^2 x}{\pi-2x}\right)}=$$

$$=e^{-2\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{\cos^2 x}{\pi-2x}\right)}=e^{-2\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{2\cos x\cdot(-\sin x)}{-2}\right)}=e^0=1.$$

7. To reveal this uncertainty in the given example, Lhospital's rule must be applied n times:

$$\lim_{x\to\infty} x^n e^{-2x} = \{\ \infty \cdot 0\ \} = \lim_{x\to\infty} \frac{x^n}{e^{2x}} = \left\{\frac{\infty}{\infty}\right\} = \lim_{x\to\infty} \frac{nx^{n-1}}{2e^{2x}} = \left\{\frac{\infty}{\infty}\right\} = \frac{n}{2}\lim_{x\to\infty} \frac{x^{n-1}}{e^{2x}} = \left\{\frac{\infty}{\infty}\right\} =$$

$$= \frac{n}{2}\lim_{x\to\infty} \frac{(n-1)x^{n-2}}{2e^{2x}} = \left\{\frac{\infty}{\infty}\right\} = \frac{n(n-1)}{2^2}\lim_{x\to\infty} \frac{x^{n-2}}{e^{2x}} = \left\{\frac{\infty}{\infty}\right\} = \frac{n(n-1)}{2^2}\lim_{x\to\infty} \frac{(n-2)x^{n-3}}{2e^{2x}} =$$

$$= \left\{\frac{\infty}{\infty}\right\} = \frac{n(n-1)(n-2)}{2^3}\lim_{x\to\infty} \frac{x^{n-3}}{e^{2x}} = \left\{\frac{\infty}{\infty}\right\} = \cdots = \frac{n!}{2^n}\lim_{x\to\infty} \frac{1}{e^{2x}} = \frac{n!}{2^n}\cdot 0 = 0 .$$

8. Find the limit $\lim\limits_{x\to 0} \dfrac{7^{x^2} - 2^{x^2}}{\left(7^x - 2^x\right)^2}$.

The solution. We have classical uncertainty $\{0/0\}$, so we immediately use Lhospital's rule twice:

$$\lim_{x\to 0} \frac{7^{x^2} - 2^{x^2}}{\left(7^x - 2^x\right)^2} = \{0/0\} = \lim_{x\to 0} \frac{7^{x^2}\cdot \ln 7 \cdot 2x - 2^{x^2}\cdot \ln 2\cdot 2x}{2\left(7^x\cdot \ln 7 - 2^x \cdot \ln 2\right)\cdot\left(7^x - 2^x\right)} = \left| \begin{array}{l} 7^{x^2}\ln 7 - 2^{x^2}\ln 2 \to \ln(7/2) \\ 7^x \ln 7 - 2^x \ln 2 \to \ln(7/2) \\ \quad if\ \ x\to 0 \end{array} \right| =$$

$$= \frac{1}{\ln(7/2)}\cdot \lim_{x\to 0} \frac{(7^{x^2}\cdot \ln 7 - 2^{x^2}\cdot \ln 2)\cdot x}{\left(7^x - 2^x\right)} = \lim_{x\to 0}\frac{x}{\left(7^x - 2^x\right)} = \lim_{x\to 0}\frac{1}{\left(7^x\cdot \ln 7 - 2^x\cdot \ln 2\right)} = \frac{1}{\ln(7/2)} .$$

When finding this limit, the results of ***Remark 8.4*** were used.

8.3. General theorems related to the representation of functions by Taylor and Maclaurin formulas

We present Taylor's formula for a function $f(x)$ with a residual term in Lagrange form (***Joseph-Louis Lagrange*** (1735–1813)):

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 +$$
$$+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},\ \ c \in (x_0, x). \qquad (8.5)$$

Remark. This form of the residual term is most often used in applied problems and scientific research. The residual term (8.5) in the Lagrange form resembles the next, regular term of the Taylor formula, only $(n+1)-\text{th}$ order the derivative of the function is calculated not at the point x_0, but at some intermediate point $c = x_0 + \theta(x - x_0)$ between the points x_0 and x.

The conditions under which a function can be represented by Taylor's formula are stated below in two central theorems.

Theorem on Taylor's local formula. *Let*:

1) *the function* $f(x)$ *is defined in some neighborhood* $|x - x_0| < \varepsilon$ *of the point* x_0;

2) $f(x)$ *has in this neighborhood derivatives* $f'(x),\ f''(x), \cdots,\ f^{(n-1)}(x)$ *up to and including the* $(n-1)$ *– th order*;

3) *there is a* n *– th order derivative* $f^{(n)}(x_0)$ *at the point* x_0.

Then at this point the local Taylor's formula is valid (with a residual term ***in Peano form***)

$$f(x) = \sum_{k=0}^{n} a_k (x - x_0)^k + o(x - x_0)^n, \tag{8.6}$$

where $a_k = \dfrac{f^{(k)}(x_0)}{k!},\ k = \overline{0,n};\ f^{(0)}(x) = f(x);\ 0! = 1$.

Under the conditions specified in the Theorem, this representation of the function is unique.

Theorem on Taylor's formula on a line segment $[a;b]$. *Let*:

1) *the function* $f(x)$ *is defined on some segment* $[a;b]$;

2) $f(x)$ *has continuous derivatives* $f'(x),\ f''(x), \cdots,\ f^{(n)}(x)$ *up to and including the* n *– th order on this segment*;

3) *for arbitrary* $x \in (a;b)$ *there is a finite derivative of* $(n+1)-$ *th order* $f^{(n+1)}(x)$.

Then the Taylor's formula (with the residual term in the form of Lagrange) is valid on the segment $[a;b]$.

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + R_{n+1}(x), \quad a \le x \le b, \tag{8.7}$$

where $R_{n+1}(x) = \dfrac{f^{(n+1)}(a + \theta(x - a))}{(n+1)!}(x - a)^{n+1},\ 0 < \theta < 1$.

If we put $x_0 = 0$ in Taylor's formula (8.5), we get a formula called Maclaurin's formula. It will have a simpler form if we use the residual term in the Lagrange form:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}, \tag{8.8}$$

where the point c is contained between the points 0 and x ($c = \theta x,\ 0 < \theta < 1$).

8.4. The technique of representing functions by Taylor–Maclaurin formulas

Let's find Maclaurin's formulas for some elementary functions.

Example 8.1. Write down Maclaurin's formula for the function $\ln(1+x)$.

The solution. Let's find the value of this function and its derivatives at the point $x=0$:

$$f(x)=\ln(1+x)\Rightarrow f(0)=0;\ f'(x)=\frac{1}{1+x}\Rightarrow f'(0)=1;$$

$$f''(x)=-\frac{1}{(1+x)^2}\Rightarrow f''(0)=-1=-1!;\ f'''(x)=\frac{2}{(1+x)^3}\Rightarrow f'''(0)=2=1\cdot 2=2!;$$

$$f^{(4)}(x)=-\frac{6}{(1+x)^4}\Rightarrow f^{(4)}(0)=-6=-1\cdot 2\cdot 3=-3!;$$

$$f^{(5)}(x)=\frac{24}{(1+x)^5}\Rightarrow f^{(5)}(0)=24=1\cdot 2\cdot 3\cdot 4=4!;$$

....

$$f^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}\Rightarrow f^{(n)}(0)=(-1)^{n-1}(n-1)!;$$

$$f^{(n+1)}(x)=\frac{(-1)^n n!}{(1+x)^{n+1}}\Rightarrow R_{n+1}(x)=\frac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}}.$$

After substituting the values of the found derivatives into Maclaurin's formula, we get:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n-1}x^n}{n}+\frac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}},\quad c=\theta x,\ 0<\theta<1.$$

Example 8.2. Write down Maclaurin's formula for the function $f(x)=e^x$.

The solution. Let's find the value of this function and its derivatives at a point $x=0$

$$f^{(k)}(x)=e^x,\ k=\overline{1,n}.$$

Now we can write Maclaurin's formula for the function $f(x)=e^x$:

$$f(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\frac{e^c}{(n+1)!}x^{n+1}.$$

Example 8.3. Write down Maclaurin's formula for the function $f(x) = \sin x$.

The solution. Let's find the value of this function and its derivatives at a point $x = 0$

$$f'(0) = 1,\ f''(0) = 0,\ f'''(0) = -1,\ f^{(4)}(0) = 0,\ f^{(5)}(0) = 1,\ \ldots,$$

$$f^{(n)}(0) = \begin{cases} 1,\ if\ \ n = 1,5,9,\cdots,4k-3,\cdots; \\ -1,\ if\ \ n = 3,7,11,\cdots,4k-1,\cdots; \quad k = 1,2,3,\cdots; \\ 0,\ if\ \ n = 2,4,6,8,\cdots,2k,\cdots. \end{cases}$$

Now let's write the Maclaurin's formula for the given function, in which only odd terms will remain (here the remaining term is given in Peano form):

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + o(x^{2n}).$$

Maclaurin's formula has a similar form for the function $f(x) = \cos x$, with the only difference that only even terms will remain in it:

$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1}).$$

The last formula can be easily obtained by differentiating the previous Maclaurin's series.

Example 8.4. Write the Maclaurin's formula for the function $f(x) = e^{\sin x}$ with precision to members x^3.

The solution. Since we have already received the representation of the exponent in the form of Maclaurin's formula, we can also represent the given function using it:

$$e^{\sin x} = 1 + \frac{\sin x}{1!} + \frac{\sin^2 x}{2!} + \frac{\sin^3 x}{3!} + o(x^3).$$

Here, instead of $o(x^3)$, we should write $o(\sin^3 x)$, but due to the equivalence of infinitesimals x and $\sin x$ we can write $o(x^3)$. Now let's use the representation of the function $\sin x$ in the form of Maclaurin's formula in the following form:

$$\sin x = x - \frac{x^3}{3!} + o(x^4).$$

Let's collect all previous representations into one expression. As a result, we get:

$$e^{\sin x} = 1 + \left(x - \frac{x^3}{6}\right) + \frac{x^2}{2} + \frac{x^3}{3!} + o(x^3).$$

In this representation, the term with x^3 disappears, so we finally get:

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + o(x^3)\,.$$

Similarly, you can get Maclaurin's formula for the function $f(x) = e^{\operatorname{tg} x}$ with accuracy $o(x^6)$:

$$f(x) = e^{\operatorname{tg} x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{3x^4}{8} + \frac{37x^5}{120} + o(x^6)\,.$$

Important Note. The Taylor and Maclaurin formulas are a ***powerful tool*** for unraveling complex uncertainties (along with Lhospital's rule). However, one should use this tool very carefully, understanding how many members of the representation need to be taken into account in the Taylor (Maclaurin) formula. Let's consider three complex examples on this topic.

Example 8.5. Find the representation of the function $f(x) = -\dfrac{2x}{(1+x^2)^2}$ by Maclaurin's formula.

The solution. It is not difficult to see that the given function is the second derivative of the elementary function $g(x) = \operatorname{arctg} x$ for which the Maclaurin's formula representation is known, namely:

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=1}^{m} (-1)^{n+1} \frac{x^{2n-1}}{2n-1} + o(x^{2m-1})\,.$$

Let's find the second derivative of the function $g(x) = \operatorname{arctg} x$, which will give the final answer for this example:

$$\left(\operatorname{arctg} x\right)'' = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \right)'' = -2x + 4x^3 - 6x^5 + \cdots = 2\sum_{n=1}^{m} (-1)^n n x^{2n-1} + o(x^{2m-1})$$

So, the Maclaurin's formula sought for the representation of the function $g(x)$ has the form:

$$f(x) = -\frac{2x}{(1+x^2)^2} = 2\sum_{n=1}^{m} (-1)^n n x^{2n-1} + o(x^{2m-1})\,.$$

8.5. Application of the representation of functions by Taylor's–Maclaurin's formulas for the calculation limits

Important Note. The Taylor and Maclaurin formulas are a ***powerful tool*** for unraveling complex uncertainties (along with Lhospital's rule). However, one should use this tool very carefully, understanding how many members of the representation need to be taken into account in the Taylor (Maclaurin) formula. Let's consider three complex examples on this topic.

Example 8.6. Find the limit: $\lim\limits_{x\to\infty}\left[x - x^2\ln\left(1+\frac{1}{x}\right)\right]$.

The solution. We have non-classical uncertainty $\{\infty-\infty\}$. Let's write Taylor's formula for the function $\ln\left(1+\frac{1}{x}\right)$:

$$\ln\left(1+\frac{1}{x}\right) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \cdots$$

$$\lim_{x\to\infty}\left[x - x^2\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \cdots\right)\right] = \lim_{x\to\infty}\left[x - x + \frac{1}{2} - \frac{1}{3x} + \cdots\right] = \lim_{x\to\infty}\left[\frac{1}{2} - \frac{1}{3x} + \cdots\right] = \frac{1}{2}.$$

Example 8.7. Find the limit: $\lim\limits_{x\to 0}\left(\frac{2+\cos x}{x^3\sin x} - \frac{3}{x^4}\right)$.

The solution. We have non-classical uncertainty $\{\infty-\infty\}$. We use Maclaurin's formulas for the functions $\sin x, \cos x$:

$$\lim_{x\to 0}\left(\frac{2+\cos x}{x^3\sin x} - \frac{3}{x^4}\right) = \lim_{x\to 0}\left(\frac{2+1-\frac{x^2}{2}+\frac{x^4}{24}-\cdots}{x^3\cdot\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\right)} - \frac{3}{x^4}\right) =$$

$$= \lim_{x\to 0}\left(\frac{\left(3-\frac{x^2}{2}+\frac{x^4}{24}-\cdots\right) - 3\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\cdots\right)}{x^4\cdot\left(1-\frac{x^2}{6}+\frac{x^4}{120}-\cdots\right)}\right) = \lim_{x\to 0}\left(\frac{x^4\left(\frac{1}{24}-\frac{1}{40}\right)}{x^4}\right) = \frac{1}{24} - \frac{1}{40} = \frac{1}{60}.$$

Example 8.8. Find the limit: $\lim\limits_{x\to 0}\left[\frac{(1+x)e^{-x} - (1-x)e^{x}}{\arcsin x - \operatorname{arctg} x}\right]$.

The solution. We have classical uncertainty $\left\{\frac{0}{0}\right\}$. We use Maclaurin's formulas for functions $\arcsin x, \operatorname{arctg} x, e^x$:

$$\lim_{x\to 0}\left[\frac{(1+x)e^{-x}-(1-x)e^{x}}{\arcsin x-\operatorname{arctg} x}\right]=\left|\begin{array}{l}\arcsin x\sim x+\frac{x^3}{6}+\frac{3x^5}{40}+\cdots, if\ \ x\to 0,\\ \operatorname{arctg} x\sim x-\frac{x^3}{3}+\frac{x^5}{5}+\cdots, if\ \ x\to 0,\\ e^x\sim 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots, if\ \ x\to 0,\\ e^{-x}\sim 1-x+\frac{x^2}{2}-\frac{x^3}{6}+\cdots, if\ \ x\to 0\end{array}\right|=$$

$$=\lim_{x\to 0}\left[\frac{(1+x)\left(1-x+\frac{x^2}{2}-\cdots\right)-(1-x)\left(1+x+\frac{x^2}{2}+\cdots\right)}{\left(x+\frac{x^3}{6}+\frac{3x^5}{40}+\cdots\right)-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)}\right]=$$

$$=\lim_{x\to 0}\frac{\left(1-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{8}+\ldots\right)-\left(1-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{8}-\ldots\right)}{\left(\frac{x^3}{2}-\frac{x^5}{8}+\frac{3x^7}{16}-\ldots\right)}=\lim_{x\to 0}\frac{\frac{2x^3}{3}+\ldots}{\frac{x^3}{2}+\ldots}=\frac{4}{3}.$$

Tasks for classroom and independent work

I. Find the given limits using the Lhospital–Bernoulli rule:

8.1. $\lim\limits_{x\to 0}\frac{3tg\,4x-12tgx}{3\sin 4x-12\sin x}$.

8.2. $\lim\limits_{x\to 0}\frac{xctgx-1}{x^2}$.

8.3. $\lim\limits_{x\to 0}\frac{x(e^x+1)-2(e^x-1)}{x^3}$.

8.4. $\lim\limits_{x\to 0}\frac{\arcsin 2x-2\arcsin x}{4x^3}$.

8.5. $\lim\limits_{x\to 1}\frac{x-x^x}{\ln x-x+1}$.

8.6. $\lim\limits_{x\to 0}\frac{\cos x-\cos(\sin x)}{x^4}$.

8.7. $\lim\limits_{x\to 1}(2-x)^{tg\frac{\pi x}{2}}$.

8.8. $\lim\limits_{x\to 0}(ctgx)^{\sin x}$.

8.9. $\lim\limits_{x\to\infty}\left(tg\,\frac{\pi x}{2x+1}\right)^{\frac{1}{x}}$.

8.10. $\lim\limits_{x\to 0}\left(\frac{a^x-x\ln a}{b^x-x\ln b}\right)^{\frac{1}{x^2}}$.

8.11. $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$. **8.12.** $\lim\limits_{x\to +\infty}\left(\dfrac{2}{\pi}arctgx\right)^{x}$.

8.13. $\lim\limits_{x\to 0}\left(\dfrac{\arcsin x}{x}\right)^{\frac{1}{x^2}}$. **8.14.** $\lim\limits_{x\to 0}\left(\dfrac{tgx}{x}\right)^{\frac{1}{x^2}}$.

II. Write down Maclaurin's formulas for the following functions:

8.15. a) $f(x)=(1+x)^{\alpha}$; b) $f(x)=\operatorname{tg} x$; c) $f(x)=\operatorname{sh} x$; d) $y=\dfrac{1}{x^2+3x+2}$; e) $f(x)=\operatorname{arctg} x$; f) $f(x)=\operatorname{ch} x$; g) $y=\ln(\sqrt{x}+1)$.

III. Write down Taylor's formulas for the following functions:

8.16. a) $y=x^2\ln(x)$ (at the point $x=1$); b) $y=\dfrac{1}{\sin(x)}$ (at the point $x=\dfrac{\pi}{2}$); c) $y=e^{\frac{1}{x}}$ (at the point $x=1$); d) $y=e^{\frac{1}{\sin x}}$ (at the point $x=\dfrac{\pi}{2}$).

IV. For the function $y=f(x)$ write down the Maclaurin's formula with residuals term in the Peano form up to x^n if:

8.17. $f(x)=\sqrt{1-2x+x^3}-\sqrt[3]{1-3x+x^2}$, $n=3$. **8.18.** $f(x)=e^{2x-x^2}$, $n=5$.

8.19. $f(x)=\ln(\cos x)$, $n=6$. **8.20.** $f(x)=\sin(\sin x)$, $n=3$.

8.21. Using Maclaurin's formula, calculate with accuracy $\varepsilon=10^{-5}$: a) $\sqrt{e}$; b) $(1,1)^{1,2}$.

V. Find out the origin and estimate the errors of approximate equalities:

8.22. $\cos x\approx 1-\dfrac{x^2}{2}+\dfrac{x^4}{24}$, $x\in[0;\pi/8]$.

8.23. $\ln(1+x)\approx x-\dfrac{x^2}{2}+\dfrac{x^3}{3}$, $x\in[0;0,2]$.

8.24. $arctgx\approx x-\dfrac{x^3}{3}+\dfrac{x^5}{5}$, $x\in[0;0,3]$.

VI. Using Maclaurin's formula, calculate the given limits:

8.25. $\lim\limits_{x\to 0}\left[\dfrac{e^{-\frac{x^2}{2}}-\cos x}{x^3\sin x}\right]$. **8.26.** $\lim\limits_{x\to 0}\dfrac{e^{x}\sin x-x(1+x)}{x^3}$.

8.27. $\lim\limits_{x\to 0}\left[\dfrac{\sin(\sin x)-x\cdot\sqrt[3]{1-x^2}}{x^5}\right]$. **8.28.** $\lim\limits_{x\to 0}\left[\dfrac{1}{x^2}-\dfrac{\operatorname{ctg} x}{x}\right]$.

8.29. $\lim\limits_{x\to 0}\left[\dfrac{\operatorname{sh}(\operatorname{tg} x)-x}{\arcsin^3 x}\right]$. **8.30.** $\lim\limits_{x\to 0}\left[\dfrac{1-(\cos x)^{\sin x}}{\operatorname{tg}^3 x}\right]$.

8.31. $\lim\limits_{x\to 0}\left[\dfrac{1-\sqrt{1+x^2}\cdot\cos x}{\operatorname{tg}^4 x}\right]$. **8.32.** $\lim\limits_{x\to 0}\left(\cos x+\dfrac{x^2}{2}\right)^{\frac{1}{x[\sin(x)-x]}}$.

VII. Find the given limits using the Lhospital–Bernoulli rule and compare with the results of the examples 4.7, 4.8, 4.13, 4.14 Practical Lesson 4:

8.33. $\lim\limits_{x\to 0}\left(\dfrac{(1+x)^n-1}{x}\right)$.

8.34. $\lim\limits_{x\to 0}\left(\dfrac{\sqrt[m]{1+x}-1-\dfrac{x}{m}}{x^2}\right)$.

8.35. $\lim\limits_{x\to 81}\dfrac{x^n-81^n-n\cdot 81^{n-1}(x-81)}{(x-81)^2}$, $n\in\mathbb{N}$.

8.36. $\lim\limits_{x\to 0}\dfrac{81^{x+p}+81^{p-x}-2\cdot 81^p}{x^2}$, $p\in\mathbb{R}$.

Answers and instructions

8.1. -2. **8.2.** $-1/3$. **8.3.** $1/6$. **8.4.** $1/4$. **8.5.** 2.

8.6. $-1/6$. **8.7.** $e^{2/\pi}$. **8.8.** 1. **8.9.** 1.

8.10. $e^{(\ln^2 a-\ln^2 b)/2}$. **8.11.** $1/2$. **8.12.** $e^{-2/\pi}$.

8.13. $e^{1/6}$. **8.14.** $e^{1/3}$. **8.17.** $\dfrac{x^2}{6}+x^3+o(x^3)$.

8.18. $1+2x+x^2-\dfrac{2}{3}x^3-\dfrac{5}{6}x^4-\dfrac{1}{15}x^5+o(x^5)$. **8.19.** $-\dfrac{x^2}{2}-\dfrac{x^4}{12}-\dfrac{x^6}{45}+o(x^6)$.

8.20. $x-\dfrac{x^3}{3}+o(x^3)$. **8.21.** a). $1{,}64872$; b). $1{,}12117$.

8.25. $1/12$. **8.26.** $1/3$. **8.27.** $19/90$. **8.28.** $1/3$.

8.29. $1/2$. **8.30.** $1/2$. **8.31.** $1/3$. **8.32.** $e^{-\frac{1}{4}}$.

8.33. n. **8.34.** $\dfrac{1-m}{2m^2}$ **8.35.** $\dfrac{n(n-1)}{2}81^{n-2}$. **8.36.** $81^p\ln^2 81$.

PRACTICAL LESSON 9
Local extremum of FOV.
The biggest and the smallest the value
of the function on the segment

9.1. The concept and definition of the local extremum of the FOV. Necessary and sufficient conditions for the existence of a local extremum of the FOV

First, we will present some important concepts and definitions related to the topic of the practical session.

Definition 9.1. *A point* x_0 *is called a point of a local maximum (or minimum) of a function* $f(x)$ *if in the domain of the function there is such a neighborhood* $0 < |x - x_0| < \delta$ *of the point* x_0 *in which the inequality holds* $f(x) < f(x_0)$ *(or* $f(x) > f(x_0)$*).*

Definition 9.2. *The points of local maximum and local minimum are called points of local extremum, and the values of the function at these points are called* ***local maximum*** *and* ***local minimum or local extremum****, respectively*.

Definition 9.3. *Points where the derivative* $f'(x)$ *is zero are called* ***stationary points***.

Definition 9.4. *The points at which the derivative* $f'(x)$ *is zero or does not exist are called critical points, or* ***critical points of the first kind***.

We formulate the conditions for the existence of a local extremum.

Theorem 9.1. (*A necessary condition for the existence of a local extremum*). *If the function* $f(x)$ *has a local extremum at a point* x_0 *and is differentiable at this point, then the condition* $f'(x_0) = 0$ *is necessarily fulfilled.*

Theorem 9.2. (*The first sufficient condition for the existence of a local extremum*).

Let x_0 *is the critical point of the first kind of function* $f(x)$*, which is continuous at this point, and let there be a neighborhood* $(x_0 - \delta; x_0 + \delta)$ *of the point* x_0*, in which the function has a derivative* $f'(x)$ *except, possibly, the point* x_0*, then*:

1) *if the derivative is* $f'(x) > 0$ *in the interval* $(x_0 - \delta; x_0)$*, and the derivative is* $f'(x) < 0$ *in the interval* $(x_0; x_0 + \delta)$*, then* x_0 *is the point of the local maximum of the function* $f(x)$;

2) *if the derivative is* $f'(x)<0$ *in the interval* $(x_0-\delta;x_0)$, *and the derivative is* $f'(x)>0$ *in the interval* $(x_0;x_0+\delta)$, *then* x_0 *is the point of the local minimum of the function* $f(x)$;

3) *if in both intervals* $(x_0-\delta;x_0)$ *and* $(x_0;x_0+\delta)$ *the derivative* $f'(x)$ *has the same sign, then* x_0 *is not an extreme point of the function* $f(x)$.

Theorem 9.3. (*The second sufficient condition for the existence of a local extremum*). *Let* x_0 *– is a stationary point of the function* $f(x)$ *(i.e.* $f'(x_0)=0$*), and there is a second continuous derivative around the point* x_0, *and* $f''(x_0)\neq 0$. *If* $f''(x_0)>0$, *then* x_0 *– is the point of the local minimum; if* $f''(x_0)<0$, *then* x_0 *– is the point of the local maximum.*

Theorem 9.4. (*The third sufficient condition for the existence of a local extremum*). *Let there exist a continuous derivative* $f^{(n)}(x)$ *in the neighborhood of a stationary point* x_0, *moreover* $f^{(n)}(x_0)\neq 0$, *and* $f'(x_0)=f''(x_0)=\cdots=f^{(n-1)}(x_0)=0$. *Then*:

1) *if* n *– is even number, and* $f^{(n)}(x_0)<0$ *then* $f(x)$ *has a local maximum at the point* x_0;

2) *if* n *– is even number, and* $f^{(n)}(x_0)>0$ *then* $f(x)$ *has a local minimum at the point* x_0;

3) *if* n *– is odd number, then* $f(x)$ *there is no local extremum at the point* x_0.

9.2. The algorithm for studying the function at the extremum. Examples of the study of functions for a local extremum

***Let us present the algorithm for studying the function at the extremum, justified by Theorem 9.1 and Theorem 9.2*:**

1) find critical points of the first kind for $f(x)$. To do this, you need to solve the equation $f'(x)=0$ and among its solutions, select only those valid points that are internal points of the domain of the function; in addition, find points where the derivative $f'(x)$ does not exist;

2) if the function $f(x)$ does not have critical points, then it does not have extrema either. If there are critical points, then it is necessary to investigate the sign of the derivative in each of the intervals into which the domain of the

function is divided by these critical points. For this, it is enough to determine the sign of the derivative at any one point of the interval, since the derivative can change its sign only when passing through the critical point;

3) by changing the sign $f'(x)$ when passing through the critical points from left to right, determine the points of maxima and minima and calculate the value of the function $f(x)$ at these points. It is advisable to summarize the research results in a table.

Let us consider a number of examples of the study of functions for a ***local extremum***.

Example 9.1. Find the critical points of the function $y = 6x - 9 - 9(x-1)^{2/3}$.

The solution. Function definition domain: $x \in (-\infty; +\infty)$.

Let's find the derivative $y'(x)$: $y'(x) = 6(1 - (x-1)^{-1/3})$; $y'(x) = 0$ at a point $x = 2$ and does not exist at a point $x = 1$. Therefore, this function has two critical points of the first kind: $x_1 = 1$, $x_2 = 2$.

Example 9.2. Find local extrema of the function $y(x) = 6x - 9 - 9(x-1)^{2/3}$.

The solution. We conduct research according to the following algorithm:

1. Function definition domain: $x \in (-\infty; +\infty)$.

2. Critical points: $x_1 = 1,\ x_2 = 2$ (see example 9.1).

3. If $x \in (-\infty; 1)$, then $f'(0) > 0$ and the function also increases on this interval. If $x \in (1; 2)$, then $f'(1{,}008) < 0$ and the function also decreases on this interval. If $x \in (2; +\infty)$, then $f'(9) > 0$ and the function increases again on the specified interval. We summarize these results in Table 9.1.

Table 9.1

x	$x \in (-\infty; 1)$	$x = 1$	$x \in (1; 2)$	$x = 2$	$x \in (2; +\infty)$
$f'(x)$	+	Does not exist	−	0	+
$f(x)$	↑	$f_{\max}(1) = -3$	↓	$f_{\min}(2) = -6$	↑

Conclusion. Therefore, the function $f(x) = 6x - 9 - 9\sqrt[3]{(x-1)^2}$ has a maximum $f_{\max}(1) = -3$ at the point $x_1 = 1$, and a minimum $f_{\max}(1) = -3$ at the point $x_2 = 2$.

Example 9.3. Find local extrema of a function

$$f(x) = \frac{x^3}{3} - \frac{3}{2}x^2 + 2x - 3.$$

The solution. Function definition domain $x \in (-\infty;+\infty)$. The derivative has the following form: $f'(x) = x^2 - 3x + 2$. We solve the equation $f'(x) = 0$: $x^2 - 3x + 2 = 0 \Rightarrow x_1 = 1$, $x_2 = 2$. From here we get ***stationary points***: $x_1 = 1$, $x_2 = 2$. There are no points where the derivative $f'(x)$ does not exist. Therefore, the stationary points are the only critical points of the given function, so it is possible to find the extrema under the second sufficient condition: since $f''(x) = 2x - 3$ and $f''(1) < 0$, when $x_1 = 1$ is also a point of a local maximum $f_{\max}(1) = -\dfrac{13}{6}$; $f''(2) > 0$, when $x_2 = 2$ – a point of a local minimum, $f_{\min}(2) = -\dfrac{7}{3}$.

Remark. From this we can conclude that the study of the function at the extremum under the second sufficient condition is ***easier*** than under the first. However, this sufficient condition can be applied to a ***narrower class*** of functions. It cannot be used to study those critical points at which the ***first derivative does not exist***, as well as those stationary points at which the second derivative is zero.

Example 9.4. Investigate the extremum of a function $f(x) = e^x + e^{-x} + 2\cos x$ at a point $x = 0$.

The solution. This function is defined at $\forall\ x \in (-\infty,+\infty)$. Let's find the corresponding derivatives:

$f'(x) = e^x - e^{-x} - 2\sin x$, $f'(0) = 0$;

$f''(x) = e^x + e^{-x} - 2\cos x$, $f''(0) = 0$;

$f'''(x) = e^x - e^{-x} + 2\sin x$, $f'''(0) = 0$;

$f^{(4)}(x) = e^x + e^{-x} + 2\cos x$, $f^{(4)}(0) = 4 > 0$.

Conclusion. A given function has a local minimum at a point.

Example 9.5. Find the critical points and investigate the extremum of the function:

$$f(x) = \frac{2x}{x^2+1}.$$

The solution. The domain of this function is the entire numerical axis: $x \in (-\infty,+\infty)$. Let's find the stationary (critical) points of this function. To do this, we define the derivative and equate it to zero:

$$f'(x) = \left(\frac{2x}{x^2+1}\right)' = 2\cdot\frac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow x_{1,2} = \pm 1.$$

So, we have only two stationary points $x_{1,2} = \pm 1$. This function has no other critical points of the first kind.

Let's examine these points for extrema, using the first sufficient condition for the existence of a local extremum (see Theorem 9.2). Let's summarize all the research results in Table 9.2.

Table 9.2

x	$x \in (-\infty;-1)$	$x = -1$	$x \in (-1;1)$	$x = 1$	$x \in (1;+\infty)$
$f'(x)$	$-$	0	$+$	0	$-$
$f(x)$	$\downarrow$	$f_{\min}(-1) = -1$	$\uparrow$	$f_{\max}(1) = 1$	$\downarrow$

***Conclusion*:** the given function has a local minimum at the point $x_1 = -1$ equal to $f_{\min}(-1) = -1$, and a local maximum at the point $x_2 = 1$ equal to $f_{\min}(1) = 1$.

Example 9.6. Find the critical points and investigate the extremum of the function:

$$f(x) = (x-2)^{\frac{2}{3}} \cdot (2x+1).$$

***The solution*.** The domain of this function is the entire numerical axis: $x \in (-\infty, +\infty)$.

Let's find the critical points of this function. To do this, we define the derivative and equate it to zero:

$$f'(x) = \left((x-2)^{\frac{2}{3}} \cdot (2x+1) \right)' = \frac{10}{3} \cdot \frac{x-1}{\sqrt[3]{x-2}} = 0.$$

So, we have only two critical points: $x_1 = 1$ (the derivative is zero – a stationary point), and $x_2 = 2$ (the derivative does not exist). We omit the research process, and we summarize the research results themselves in Table 9.3.

Table 9.3

x	$x \in (-\infty;1)$	$x = 1$	$x \in (1;2)$	$x = 2$	$x \in (2;+\infty)$
$f'(x)$	$+$	0	$-$	does not exist	$+$
$f(x)$	$\uparrow$	$f_{\max}(1) = 3$	$\downarrow$	$f_{\min}(2) = 0$	$\uparrow$

***Conclusion*:** the given function has a local maximum at a point $x_1 = 1$ equal to $f_{\max}(1) = 3$, and a local minimum at a point $x_2 = 2$ equal to $f_{\min}(2) = 0$.

9.3. The largest and smallest value of the function on the segment. Examples of finding the largest and smallest values of a function on a segment

Let’s formulate an algorithm for finding the largest (smallest) value of a function $f(x)$ that is continuous on a segment $[a;b]$:

1. Find the critical points of the first kind of function $f(x)$, which belong to the interval $(a;b)$.

2. Calculate the value of the function $f(x)$ at the found critical points and limit points segment $x_0 = a$, $x_0 = b$ and then choose the largest (smallest) among these values.

Let’s consider a numbthener of examples of finding ***the largest and smallest values*** of a ***function on a segment***.

Example 9.7. Find the largest and smallest value of the function $f(x) = 3x - x^3$ on the segment $[-2,3]$.

The solution. We find the first derivative $f'(x) = (3x - x^3)' = 3 - 3x^2 = 0$ and stationary points: $x_{1,2} = \pm 1$. We determine the value of the function at stationary points:

$$f(1) = 2 \quad \text{and} \quad f(-1) = -2 .$$

Next, we calculate the value of the function at the boundaries of the segment:

$$f(-2) = 2 \quad \text{and} \quad f(3) = -18 .$$

From the obtained four values, choose the largest and smallest.

Conclusion. The largest value of the function on the given segment is $f(1) = f(-2) = 2$, and the smallest is $f(3) = -18$.

Example 9.8. Find the lengths of the sides of the rectangle with the largest area that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > 0, b > 0$ – the lengths of the corresponding semi-axes of the ellipse.

The solution. Let’s construct the objective function – the expression of the area of the rectangle. Let’s choose an arbitrary point $M_1(x, y)$ that lies on the ellipse in the first quarter of the Cartesian rectangular coordinate system XOY. It is obvious that the variables (x, y) are related to each other by the following ratio:

$$y = b\sqrt{1 - \frac{x^2}{a^2}}.$$

Let's form a rectangle with vertices at the following points: $M_1(x, y)$, $M_2(-x, y)$, $M_3(-x, -y)$, $M_4(x, -y)$. Write down the formula for the area of a rectangle $M_1M_2M_3M_4$.

$$S = 2x \cdot 2y = 4xb\sqrt{1 - \frac{x^2}{a^2}} \to \max_x .$$

So, the task was reduced to the study of the objective function $S(x) = 4xb\sqrt{1 - \frac{x^2}{a^2}}$ for a ***local maximum***.

Let's find the derivative of the function $S(x)$ and equate it to zero:

$$S'(x) = \left(4xb\sqrt{1 - \frac{x^2}{a^2}}\right)' = 4b\sqrt{1 - \frac{x^2}{a^2}} - 4xb\frac{\frac{2x}{a^2}}{2\sqrt{1 - \frac{x^2}{a^2}}} = 4b\sqrt{1 - \frac{x^2}{a^2}} - \frac{4x^2b}{a^2\sqrt{1 - \frac{x^2}{a^2}}} = 0 .$$

IWS! Find the second derivative $S''(x)$ and check what sign it has under the condition that $0 < x < a$.

After identical transformations, we get the quadratic equation:

$$1 - \frac{2x^2}{a^2} = 0,$$

which has two roots, but we choose only the positive root because $a > 0, b > 0$:

$$x = \frac{a}{\sqrt{2}}.$$

The expression for the second unknown has a similar form:

$$y = \frac{b}{\sqrt{2}}.$$

Conclusion. The sides of the rectangle $M_1M_2M_3M_4$ with the largest area inscribed in the given ellipse should be as follows: the horizontal side M_1M_2 should have the size $\sqrt{2} \cdot a$, and the vertical side M_2M_3 – $\sqrt{2} \cdot b$. The area of such a rectangle is: $S_{\max} = 2ab$.

Example 9.9. Find the dimensions of a right circular cylinder, which would have the ***largest volume*** under the condition that the area S of its complete surface is fixed and given.

The solution. Let the radius of the base of the cylinder be equal to R, and the height equal to H. Let's write down the expression for calculating the total surface area of this cylinder:

$$S = 2\pi R^2 + 2\pi RH .$$

Let's express the height H from this formula

$$H = \frac{S - 2\pi R^2}{2\pi R}.$$

Now let's construct the objective function (expression for the volume V) that needs to be maximized:

$$V = \pi R^2 H = \pi R^2 \cdot \frac{S - 2\pi R^2}{2\pi R} = \frac{SR}{2} - \pi R^3 .$$

Thus, the task was reduced to the study of the objective function $V(R) = \frac{SR}{2} - \pi R^3$ for a local extremum under the obvious condition that $R > 0$.

Let's find the derivative of the objective function $V(R)$ and set it equal to zero:

$$V'(R) = \left(\frac{SR}{2} - \pi R^3\right)' = \frac{S}{2} - 3\pi R^2 = 0 \Rightarrow R = \sqrt{\frac{S}{6\pi}} .$$

Next, we find the derivative of the second order $V''(R)$ and set its sign:

$$V''(R) = -6\pi R .$$

Since we obtained the inequality $V''(R) < 0$, the volume of the cylinder will be maximum if its dimensions are chosen as follows:

$$R = \sqrt{\frac{S}{6\pi}}, \; H = 2R = 2\sqrt{\frac{S}{6\pi}} .$$

Conclusion. The axial section of such a cylinder should be a square.

Example 9.10. Find the sides of a rectangle with the largest perimeter inscribed in a given semicircle with radius R.

The solution. Let the vertical side of the inscribed rectangle be equal to x, and half of the horizontal side – y (make a drawing yourself). Then the perimeter

P of the rectangle (as an objective function) will take the form:

$$P = 2x + 4y \to \max .$$

Now let's write the relation that connects the unknowns x and y:

$$x^2 + y^2 = R^2 .$$

Next, let's express one of the variables in terms of another $y = \sqrt{R^2 - x^2}$ and substitute its expression into the target function:

$$P(x) = 2x + 4\sqrt{R^2 - x^2} \to \max_x .$$

So, the problem is reduced to the problem of finding the local extremum of a function of one variable. After differentiation, we get the equation:

$$P'(x) = 2 + 4\frac{(-x)}{\sqrt{R^2 - x^2}} = 0 \Rightarrow 1 = \frac{2x}{\sqrt{R^2 - x^2}} .$$

Solving this equation, we have: $x = \frac{R}{\sqrt{5}}$. Let's find the second unknown from the equation: $y = \sqrt{R^2 - x^2}$. As a result, we have: $y = \frac{2R}{\sqrt{5}}$.

***Answer*:** the sides of the required rectangle should be equal to: $x = \frac{R}{\sqrt{5}}$, $2y = \frac{4R}{\sqrt{5}}$.

IWS! Find the second derivative $P''(x)$ and find out its sign.

Example 9.11. At what height above the center of a round table with a radius R should an electric lantern be placed so that the illumination of the edge of the table is the greatest?

***Indication*.** The brightness I of the lighting is described by the formula

$$I = \frac{k \sin \varphi}{h^2}, \tag{9.1}$$

where φ – is the angle of inclination of the light rays; h – the distance of the light source from the illuminated area; k – power of the light source.

***The solution*.** Let's make a picture Fig. 9.1. On it, the light source is marked with the letter A, the center of the table with the letter B, the periphery of the table with the letter C. All other designations and explanations are given below in the solution text.

Let x – the desired height of the light source be placed above the center of the table. Let's write certain relations that connect all the characteristics contained in the formula (9.1):

$$\frac{x}{h} = \sin\varphi \Rightarrow x = h\sin\varphi = R\,\mathrm{tg}\,\varphi\,,\quad h = \sqrt{x^2 + R^2}\,,\ \text{or}\ \ h = \frac{R}{\cos\varphi}.$$

Substitute the obtained ratio into formula (9.1) and find the maximum of the obtained function by variable φ:

$$I = \frac{k\sin\varphi\cos^2\varphi}{R^2} \to \max_{\varphi}\,;$$

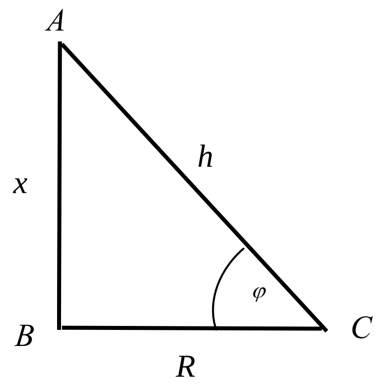

Fig. 9.1

$$\left(\frac{k\sin\varphi\cos^2\varphi}{R^2}\right)' = 0 \Rightarrow \left(\sin\varphi\cos^2\varphi\right)' = 0 \Rightarrow \mathrm{tg}\,\varphi = \frac{1}{\sqrt{2}}.$$

Now you can determine the desired distance x:

$$x = R\,\mathrm{tg}\,\varphi = \frac{R}{\sqrt{2}}.$$

Tasks for classroom and independent work

I. Find intervals of monotonicity of functions:

9.1. $y = \ln(4 - x^2)$.

9.2. $y = \dfrac{2x}{x^2 + 1}$.

II. Find local extrema of functions:

9.3. $y = x^{\frac{2}{3}} \cdot e^{-\frac{x}{3}}$.

9.4. $y = \dfrac{x^3}{1 - x}$.

9.5. $y = x\sqrt{1-x}$.   **9.6.** $y = \dfrac{(x-2)(3-x)}{x^2}$.

9.7. $y = x + \dfrac{1}{x}$.   **9.8.** $y = arctgx - \dfrac{1}{2}\ln(x^2+1)$.

III. Find the largest and smallest values of the function $f(x)$ on the segment $[a;b]$:

9.9. $y = x^4 - 2x^2 + 3,\ x \in [-3;2]$. **9.10.** $y = \dfrac{x^2+3}{x^2+2x+5},\ x \in [-1;3]$.

9.11. A trapezoid is inscribed in a semicircle with a radius R, the base of which is the diameter of the semicircle. Determine the angle at the base of the trapezoid so that its area is the largest.

9.12. A wire of length L should be cut into two parts, one of which should be bent into a circle, and the second into a square. At what length of each of the parts will the sum of the areas of the circle and the square be the largest?

9.13. Inscribe the cylinder of the largest volume into the cone with the height H and radius R of the base. What are the dimensions of this cylinder?

9.14. A weight load P lying on a horizontal rough surface needs to be moved by some applied force $\vec{F}$. At what angle of inclination φ of the vector of action of this force $\vec{F}$ to the horizon will its magnitude be the smallest? Assume that the coefficient of friction between the load and the rough surface is equal to k.

9.15. The bending resistance of a beam of rectangular section is directly proportional to the product of its base b by the square of the height h. What should be the cross-sectional dimensions of the beam, if it is cut from a round log with a diameter D under the condition that its resistance to bending would be the greatest?

9.16. It is necessary to make a box with a lid, the volume of which is equal to V. The shape of the box is a rectangular parallelepiped, and the lengths of the sides of the base are equal to one to two. What should be the dimensions of all the ribs of the box, so that the least amount of sheet material was spent on its manufacture?

9.17. ***)**A uniform rod of length $l > 2R$ is placed in a cup having the shape of a hemisphere of radius R. Find the equilibrium position of the rod under the condition that the rod can move in the cup without sliding friction. (***Reference.*** The equilibrium position of the body is achieved under the condition of a ***minimum of its potential energy***).

9.18. Enterprise P must be connected by a highway with a straight section of the railway that passes through city A. The distance from the enterprise P to the nearest point B of the railway is $|PB| = 40$ km, and the distance from the city to the same point is $|AB| = 200$ km. The cost of transportation by highway is $\sqrt{5}$

times higher then by the rail. To what point should a highway be laid from the railway so that the cost of transportation is the lowest?

9.19. What should be the angle α at the top of an isosceles triangle of a given area S, provided that the radius of the circle inscribed in this triangle would be the largest?

9.20. A right parallelepiped is inscribed in a hemisphere with radius R, based on a regular hexagon. Find the height and side of the base of the parallelepiped if its volume is maximum.

9.21.[*)] Solve ***task 9.13***, but instead of a cylinder, you need to take a straight regular polyhedral prism with the number of faces equal to $2n$.

Answers and instructions

9.1. $(-2;0)$ – increases, $(0;2)$ – decreasing.

9.2. $(-\infty;-1),\ (1;+\infty)$ – decreasing, $(-1;1)$ – increases.

9.3. $y_{\min}(0)=0,\ y_{\max}(2)=\sqrt[3]{4}e^{-2/3}$. **9.4.** $y_{\min}(3/2)=-6{,}75$.

9.5. $y_{\max}(2/3)=2\sqrt{3}/9$. **9.6.** $y_{\max}(2,4)=1/24$.

9.7. $y_{\min}(1)=2,\ y_{\max}(-1)=-2$.

9.8. $y_{\max}(1)=\pi/4-0{,}5\ln 2$. **9.9.** $y_{\min}=2,\ y_{\max}=66$.

9.10. $y_{\min}=0{,}5,\ y_{\max}=1$. **9.11.** $\dfrac{\pi}{3}$. **9.12.** $\dfrac{\pi L}{4+\pi},\ \dfrac{4L}{4+\pi}$.

9.13. Cylinder height – $h=\dfrac{H}{3}$; base radius – $r=\dfrac{2}{3}R$.

9.14. $\varphi=\operatorname{arctg}k$. **9.15.** $b=\sqrt{3}D/3,\ h=\sqrt{6}D/3$.

9.16. $h=2\sqrt[3]{\dfrac{V}{9}},\ a=\sqrt[3]{3V},\ b=\dfrac{\sqrt[3]{3V}}{2}$.

9.17. $\cos\alpha=\dfrac{l+\sqrt{l^2+128R^2}}{16R}$ at $l\le 4R$; at $l>4R$ – no equilibrium position.

9.18. $|AC|=180$ km. **9.19.** $\alpha=\dfrac{\pi}{3}$. **9.20.** $a=\sqrt{\dfrac{2}{3}}R,\ h=\dfrac{R}{\sqrt{3}}$.

9.21. The radius of the circle in which a regular polygon is inscribed is equal to $r=\dfrac{2}{3}R$; the height of the parallelepiped is $h=\dfrac{H}{3}$.

PRACTICAL LESSON 10

Convexity-concavity of the FOV, inflection points.
Asymptotes of the curve.
A complete study of functions by the methods of differential calculus

10.1. Convexity-concavity of the FOV, inflection points

As before each practical lesson, we will briefly state the necessary theoretical provisions regarding the stated topic of Practical Lesson No 10.

We will give the definition of ***convexity-concavity*** of curves with the involvement of ***derivatives of the first and second order***. Involvement of ***the derivative of the first order*** leads to the use of ***the concept of tangent*** in the definitions of convexity-concavity. Let's list them.

Definition 10.1. *A curve $y = f(x)$ is called* ***convex*** *(denoted by $\cap$) on the interval $(a;b)$, if all its points, except for the point of contact, lie* ***below*** *its arbitrary tangent on this interval.*

Definition 10.2. *A curve $y = f(x)$ is called* ***concave*** *(denoted by $\cup$) on the interval $(a;b)$, if all its points, except for the point of contact, lie* ***above*** *any of its tangents on this interval*.

Definition 3. *The* ***inflection point*** *is the point of the curve $y = f(x)$ that* ***separates its convex part from the concave part***.

The convexity and concavity intervals of the curve $y = f(x)$ are found by the results of Theorem 10.1.

Theorem 10.1. *Let the function $y = f(x)$ be twice differentiable on $(a;b)$, then*:

1) *if $f''(x) < 0, \forall\ x \in (a;b)$, then the curve $y = f(x)$ is* ***convex*** *($\cap$) on* $(a;b)$;

2) *if $f''(x) > 0, \forall\ x \in (a;b)$, then the curve $y = f(x)$ is* ***concave*** *($\cup$) on* $(a;b)$.

Definition 10.4. *The points at which the second derivative $f''(x)$ is zero or does not exist are called* ***critical points of the second kind*** *of the function* $y = f(x)$.

Therefore, if x_0 – is the abscissa of the inflection point of the function $y = f(x)$, then x_0 is the critical point of the second kind of this function. The converse is ***not always true***.

Let us establish ***sufficient conditions for the existence*** of an inflection point.

Theorem 10.2. *Let be* x_0 *– the critical point of the second kind of function* $y = f(x)$. *If the derivative* $f''(x)$ *changes sign when passing through the critical point* x_0, *then the point* $M_0\left(x_0; f(x_0)\right)$ *is an* ***inflection point*** *of the curve* $y = f(x)$.

We present the ***algorithm for finding the inflection points*** of the curve $y = f(x)$.

1) find the domain of the given function $y = f(x)$;

2) find the second derivative $f''(x)$ of this function and establish critical points of the second kind from two conditions:

a) the second derivative $f''(x)$ is zero and

b) $f''(x)$ does not exist;

3) investigate the change in the sign of the second derivative when passing through each of the critical points x_0: if the second derivative changes sign when passing through the selected critical point x_0, then the point $M_0\left(x_0; f(x_0)\right)$ is the ***inflection point*** of the function $y = f(x)$; otherwise, this point is not an inflection point.

Let's consider typical examples of ***finding convexity-concavity intervals*** and inflection points

Example 10.1. Find the convexity-concavity intervals and inflection points of the function $f(x) = xe^{-x}$.

The solution. The domain of the given function: $x \in (-\infty; +\infty)$.

The first derivative has the following form: $f'(x) = e^{-x}(1 - x)$. The second derivative looks like this: $f''(x) = e^{-x}(x - 2)$. Hence we have a single ***critical point*** of the second kind: $x = 2$. We divide the domain of definition of the function with the found point into two intervals and determine the sign of the second derivative in each of them:

1) $x \in (-\infty; 2)$; $f''(0) < 0$, the function is convex ($\cap$);

2) $x \in (2; +\infty)$; $f''(3) > 0$, the function is concave ($\cup$).

Conclusion. The point $M_0\left(2; 2e^{-2}\right)$ is the ***inflection point*** of the curve $f(x) = xe^{-x}$.

Example 10.2. Find the convexity-concavity intervals and inflection points of the function $y = x(x-5)^{\frac{2}{3}}$.

The solution. The domain of the given function: $x \in (-\infty;+\infty)$.

Let's find the first derivative: $y' = \left(x(x-5)^{\frac{2}{3}} \right)' = \frac{5}{3} \cdot \frac{x-3}{\sqrt[3]{x-5}}$.

Let's find the second derivative: $y'' = \left(\frac{5}{3} \cdot \frac{x-3}{\sqrt[3]{x-5}} \right)' = \frac{10}{9} \cdot \frac{x-6}{(x-5)^{4/3}}$.

The set of critical points of the second kind consists of two points $\{5;6\}$: when $x_1 = 5$ the second derivative does not exist; when $x_2 = 6$ the second derivative is zero.

We check the change in the sign of the second derivative y'' when passing through each of these points:

1) $x \in (-\infty;5)$, $y''(4) < 0$; therefore, the function is convex ($\cap$);

2) $x \in (5;6)$, $y''(5,5) < 0$; therefore, the function is convex ($\cap$);

3) $x \in (6;+\infty)$, $y''(7) > 0$; therefore, the function is concave ($\cup$).

Conclusion. The point $M(6,6)$ – is the only inflection point of the curve $y = x(x-5)^{\frac{2}{3}}$.

10.2. Asymptotes of the curve

Definition. *A straight line* l *is called an* ***asymptote of a curve*** *if the distance* δ *from the variable point* M *of the curve to this straight line* l *goes to zero, if the point* M *moving along the curve recedes to infinity (Fig. 10.1).*

The curve $y = f(x)$ can have ***vertical***, ***horizontal*** and ***inclined*** asymptotes.

1. A straight line $x = x_0$ is a vertical asymptote of a curve $y = f(x)$ if $\lim\limits_{x \to x_0 - 0} f(x) = \infty$, or $\lim\limits_{x \to x_0 + 0} f(x) = \infty$, or $\lim\limits_{x \to x_0} f(x) = \infty$.

2. A straight line $y = y_0$ is a horizontal asymptote of a curve $y = f(x)$ if $\lim\limits_{x \to -\infty} f(x) = y_0$, or $\lim\limits_{x \to +\infty} f(x) = y_0$, or $\lim\limits_{x \to \pm\infty} f(x) = y_0$.

3. A straight line $y = kx + b$ is an inclined asymptote of a curve $y = f(x)$ if there are finite boundaries:

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = k \neq 0, \qquad \lim_{x \to \pm\infty} \left[f(x) - kx \right] = b. \qquad (10.1)$$

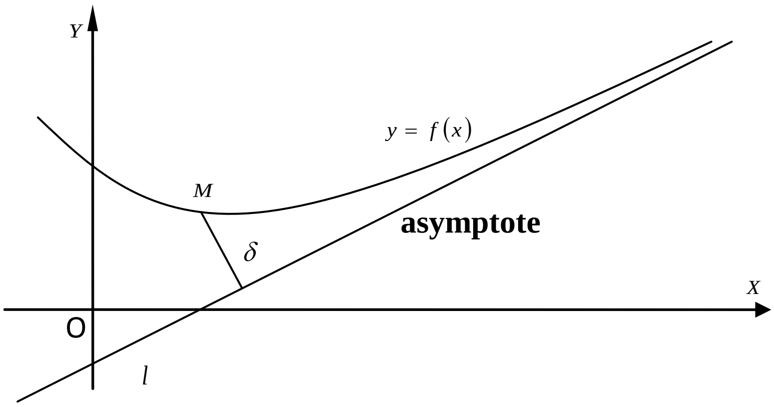

Fig. 10.1

Remark 10.1. If at least one of the limits (10.1) does not exist or is equal to infinity, then the curve $y = f(x)$ does not have a sloping asymptote.

Remark 10.2. If $k = 0$, then $b = \lim_{x \to \infty} f(x)$, therefore the equation of the horizontal asymptote has the form: $y = b$. Since this equation is a special case of the equation $y = kx + b$, there are not three, but two types of asymptotes: ***vertical and non-vertical***.

Remark 10.3. The asymptotes of the curve $y = f(x)$ at $x \to +\infty$ and $x \to -\infty$ may be different. Therefore, when finding the asymptotes of the boundary (10.1), it is necessary to calculate separately for $x \to +\infty$ and $x \to -\infty$.

Example 10.3. Find the asymptotes of the curve: $f(x) = \dfrac{3x^2 + 4x + 1}{x}$.

The solution. Let's find the vertical asymptotes. Since $f(x)$ it is not defined at point $x = 0$ and the corresponding limits are equal

$$\lim_{x \to -0} \frac{3x^2 + 4x + 1}{x} = -\infty, \quad \lim_{x \to +0} \frac{3x^2 + 4x + 1}{x} = +\infty,$$

then straight line $x = 0$ is the ***vertical asymptote*** of the curve.

We are looking for a ***inclined*** asymptote according to formulas (10.1) at $x \to +\infty$:

$$k = \lim_{x \to +\infty} \frac{3x^2 + 4x + 1}{x^2} = 3, \quad b = \lim_{x \to +\infty} \left(\frac{3x^2 + 4x + 1}{x} - 3x \right) = 4.$$

The straight line $y = 3x + 4$ is the ***inclined*** asymptote of this curve at $x \to +\infty$. It is not difficult to make sure that this line is an asymptote and at $x \to -\infty$. So, the given curve has two asymptotes: vertical $x = 0$ and inclined $y = 3x + 4$.

Example 10.4. Find the asymptotes of the curve: $y(x) = \sqrt{\frac{x^3}{x-2}}$.

The solution. The function is defined in the intervals $(-\infty, 0)$ and $(2, +\infty)$. Since the limits

$$\lim_{x \to 2+0} \sqrt{\frac{x^3}{x-2}} = +\infty ,$$

then the straight line $x = 2$ is the vertical asymptote of the curve $y(x)$.

The curve has no horizontal asymptotes, since the limits $\lim_{x \to \pm\infty} \sqrt{\frac{x^3}{x-2}} = +\infty$ are not finite values.

Let's establish the existence of inclined asymptotes.

I. Let's start with the right asymptote for $x \to +\infty$:

1) $k_1 = \lim_{x \to +\infty} \frac{y(x)}{x} = \lim_{x \to +\infty} \sqrt{\frac{x^3}{(x-2)x^2}} = 1$;

2) $b_1 = \lim_{x \to +\infty} [y(x) - k_1 x] = \lim_{x \to +\infty} \left(\sqrt{\frac{x^3}{(x-2)}} - x \right) = \lim_{x \to +\infty} \left(\frac{x(\sqrt{x} - \sqrt{x-2})}{\sqrt{x-2}} \right) =$

$$= \lim_{x \to +\infty} \left(\frac{2x}{\sqrt{x-2}(\sqrt{x} + \sqrt{x-2})} \right) = 1 .$$

Thus, the right inclined of the asymptote exists and its equation has the form: $y = x + 1$.

II. Now let's establish the existence of the left asymptote for $x \to -\infty$:

$$1)\ k_2 = \lim_{x \to -\infty} \frac{y(x)}{x} = \lim_{x \to -\infty} \frac{\sqrt{\frac{x^3}{(x-2)}}}{x} = \lim_{x \to -\infty} \frac{\sqrt{\frac{x}{(x-2)}}}{x} \cdot (-x) = \left| \begin{matrix} x < 0; \\ -x > 0 \end{matrix} \right| = -\lim_{x \to -\infty} \sqrt{\frac{x}{(x-2)}} = -1;$$

$$2)\ b_2 = \lim_{x \to -\infty} [y(x) - k_1 x] = \lim_{x \to -\infty} \left(\sqrt{\frac{x^3}{x-2}} + x \right) = \lim_{x \to -\infty} \left(\sqrt{\frac{(-x)^3}{2-x}} + x \right) =$$

$$= \lim_{x \to -\infty} \frac{-x\sqrt{-x} + x\sqrt{2-x})}{\sqrt{2-x}} = -\lim_{x \to -\infty} \left(\frac{x(\sqrt{-x} - \sqrt{2-x})}{\sqrt{2-x}} \right) =$$

$$= -\lim_{x \to -\infty} \left(\frac{-2x}{\sqrt{2-x}(\sqrt{-x} + \sqrt{2-x})} \right) = -1 .$$

Thus, the left inclined of the asymptote also exists and its equation has the form: $y = -x - 1$.

Conclusion. The curve $y(x)=\sqrt{\dfrac{x^3}{x-2}}$ has three asymptotes: vertical $x=2$, left inclined $y=-x-1$, and right inclined $y=x+1$.

Example 10.5. Find the asymptotes of the curve $y(x)=x+2\operatorname{arctg} x$.

The solution. It is easy to check that the given function has no vertical and horizontal asymptotes. Let's find the inclined asymptotes. First, let's direct $x\to+\infty$ and find the right inclined asymptote:

1) $k_1=\lim\limits_{x\to+\infty}\dfrac{x+2\operatorname{arctg} x}{x}=\lim\limits_{x\to+\infty}\left(1+2\dfrac{\operatorname{arctg} x}{x}\right)=1$;

$$b_1=\lim_{x\to+\infty}(x+2\operatorname{arctg} x-x)=2\lim_{x\to+\infty}\left(\operatorname{arctg} x\right)=2\cdot\frac{\pi}{2}=\pi .$$

The equation of the right inclined asymptote was obtained: $y=x+\pi$.

Now let's direct $x\to-\infty$ and find the left inclined asymptote;

2) $k_2=\lim\limits_{x\to-\infty}\dfrac{x+2\operatorname{arctg} x}{x}=\lim\limits_{x\to-\infty}\left(1+2\dfrac{\operatorname{arctg} x}{x}\right)=1$;

$$b_2=\lim_{x\to-\infty}(x+2\operatorname{arctg} x-x)=2\lim_{x\to-\infty}\left(\operatorname{arctg} x\right)=2\cdot\left(-\frac{\pi}{2}\right)=-\pi .$$

The equation of the left inclined asymptote was obtained: $y=x-\pi$.

Example 10.6. Find the asymptotes of the curve given ***parametrically***:

$$\begin{cases} x(t)=\dfrac{1}{t};\\ y(t)=\dfrac{t}{t+1}.\end{cases}$$

The solution. We first consider y as a function of x: $y=y(x)$. Then if the parameter t goes to zero, then x goes to infinity. From this condition, we will find the coefficients k and b (which are equal to zero):

$$k=\lim_{t\to0}\frac{y}{x}=\lim_{t\to0}\frac{t^2}{1+t}=0;$$

$$b=\lim_{t\to0}[y-kx]=\lim_{t\to0}y=0.$$

Thus, ***the first asymptote*** has the equation: $y=0$.

Now consider the given function in another form, as $x=x(y)$. Then you should find the above-mentioned limits under the condition that y goes to infinity, that is, if the parameter $t\to-1$. Let's find the coefficients k and b for this case:

$$k_1=\lim_{t\to-1}\frac{x}{y}=\lim_{t\to-1}\frac{1+t}{t^2}=0;\qquad b_1=\lim_{t\to-1}[x-k_1y]=\lim_{t\to-1}\frac{1}{t}=-1.$$

Thus, ***the second asymptote*** has the equation: $x = -1$.

Remark. It is interesting that in the given parametric equations it is possible to get rid of the parameter t and write the equation of the curve in the form $y = (1+x)^{-1}$, from which it becomes obvious that this curve has two found asymptotes.

10.3. A complete study of functions by the methods of differential calculus

Let's formulate ***the algorithm for the complete study of the function*** by the methods of differential calculus with the construction of its graph:

1) find the domain of the function;

2) find the points of intersection of the graph with the coordinate axes (if they exist);

3) investigate the function for periodicity, evenness and oddness;

4) find breakpoints and investigate the behavior of the function around them;

5) find intervals of monotonicity, points of local extrema and function values at these points;

6) find convexity-concavity intervals and inflection points;

7) find all the asymptotes of the curve;

8) on the basis of preliminary studies carried out in paragraphs 1) – 7), plot the graph of the function.

Remark. If the function is ***even*** (or ***odd***), then it is enough to plot its graph for $x \geq 0$, and then reflect it symmetrically with respect to the axis OY (or with respect to the origin O of the coordinates).

We will consider the algorithm of a complete study of functions by the methods of differential calculus on a specific example.

Example 10.7. Investigate and plot a graph of a function $y = \dfrac{(x-1)^2}{x+2}$.

The solution.

1. ***The domain*** of the given function is the entire numerical axis, except for the point $x = -2$.

2. ***Intersection*** of the graph with the axis: OY : $y = 1/2$ if $x = 0$. We have a point on the axis $OY : B\left(0, \frac{1}{2}\right)$. ***Intersection*** of the graph with the axis OX : $x = 1$ if $y = 0$. We have a point on the axis $OX : A(1, 0)$.

3. The function is ***non-periodic***. Since $f(-x) = \frac{(-x-1)^2}{-x+2} = \frac{(x+1)^2}{2-x} \neq \pm f(x)$, the given function ***has neither central nor axial symmetry***.

4. *Break points*. The point $x = -2$ is a ***discontinuity point of the second kind***, since $\lim\limits_{x \to -2 \pm 0} f(x) = \pm\infty$. Therefore, the line $x = -2$ is the ***vertical asymptote*** of the curve.

5. *Monotonicity intervals, function extrema*. Let's find the first derivative:

$$y' = \frac{2(x-1)(x+2) - (x-1)^2}{(x+2)^2} = \frac{(x+5)(x-1)}{(x+2)^2} = 1 - 9 \cdot (x+2)^{-2}.$$

Therefore, $y' = 0$ at $(x-1)(x+5) = 0 \Rightarrow x_1 = 1, x_2 = -5$ are stationary points of the function $f(x)$ ($x_3 = -2$ is a critical point that $f'(x)$ does not exist in it; moreover, this point does not belong to the domain of definition of the function $f(x)$ itself). We determine the sign of the derivative when passing through the critical points and conclude about the extrema of the function (Fig. 10.2, *a*).

***Conclusion*:** when $x = -5$, the function has a local maximum $y_{\max} = -12$, when $x = 1$ – a local minimum $y_{\min} = 0$.

6. *Convexity-concavity, inflection* points.

We find the second derivative: $y'' = 18(x+2)^{-3}$. The derivative $y'' \neq 0$ and does not exist when $x = -2$. However, the critical point of the second kind $x = -2$ does not belong to the domain of the function $f(x)$, so there are ***no inflection points***. Let's determine the sign of the second derivative in the domain of the function (Fig. 10.2, *b*).

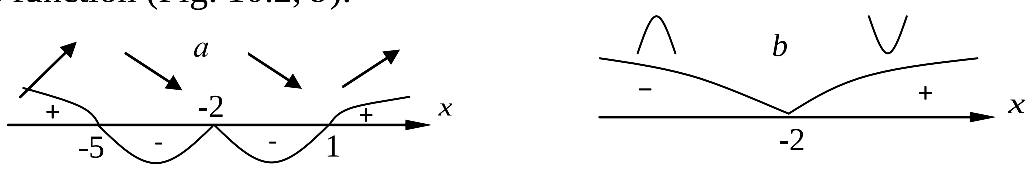

Fig. 10.2

7. *Asymptotes*. In ***clause 4***, it was established that a straight line $x = -2$ is a ***vertical asymptote*** of a curve $f(x)$. Let's find the ***inclined asymptote*** $y = kx + b$. Let's determine the numbers k and b:

$$k = \lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{(x-1)^2}{x^2 + 2x} = \left\{ \frac{\infty}{\infty} \right\} = \lim_{x \to \pm\infty} \frac{x^2\left(1 - \frac{2}{x} + \frac{1}{x^2}\right)}{x^2\left(1 + \frac{2}{x}\right)} = \lim_{x \to \pm\infty} \frac{1 - \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{2}{x}} = 1;$$

$$b = \lim_{x\to\pm\infty}\left[f(x) - kx\right] = \lim_{x\to\pm\infty}\left[\frac{(x-1)^2}{x+2} - x\right] = \{\infty - \infty\} = \lim_{x\to\pm\infty}\frac{-4x+1}{x+2} = -4.$$

Therefore, a straight line $y = x - 4$ is a ***inclined asymptote***.

8. ***Graphing the function*** $y = \dfrac{(x-1)^2}{x+2}$. Based on the research carried out in ***clauses 1 – 7*** plot the given function (Fig. 10.3).

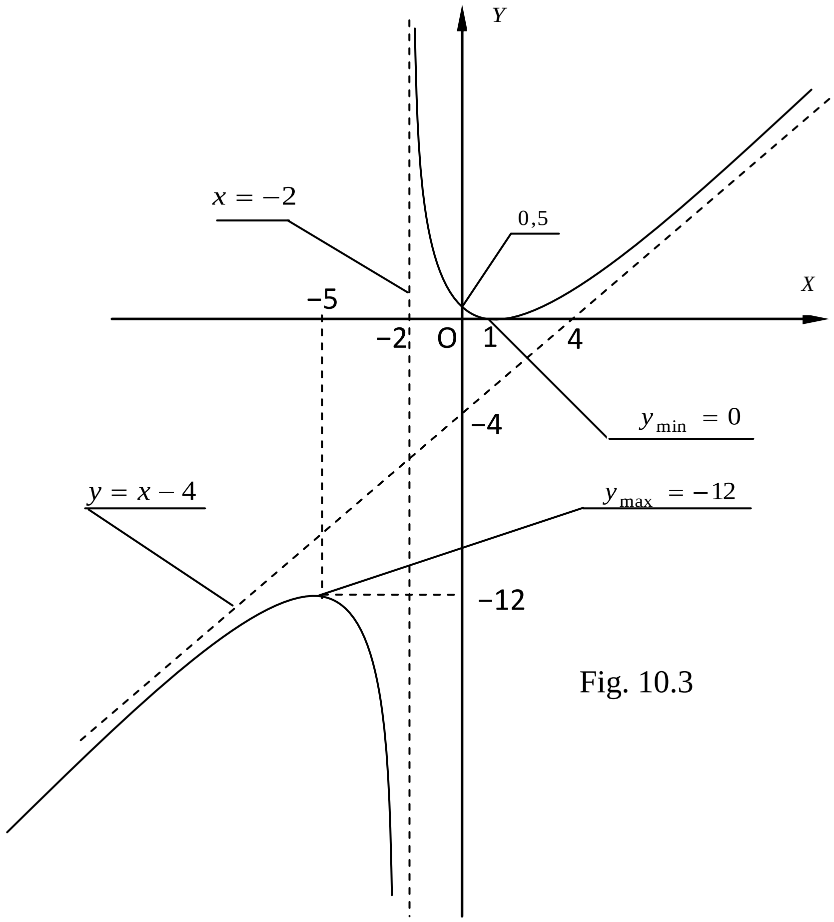

Fig. 10.3

Tasks for classroom and independent work

I. Find intervals of concavity-convexity and inflection points of the curves specified in the Cartesian coordinate system:

10.1. $f(x) = x^4 - 8x^3 + 18x^2 - 6x + 1$. **10.2.** $f(x) = xe^{-2x}$.

10.3. $f(x) = \sqrt[3]{(x^2-4)^2}$. **10.4.** $f(x) = x^4 - 4x^3 - 18x^2 + 2x - 1$.

II. Find the inflection points of the curves given parametrically and in the polar coordinate system:

10.5. $\begin{cases} x = t^2; \\ y = 3t + t^3. \end{cases}$ **10.6.** $\begin{cases} x = e^t; \\ y = \sin t. \end{cases}$ **10.7.** $\begin{cases} x = 2a\cos t - a\cos 2t; \\ y = 2a\sin t - a\sin 2t. \end{cases}$

10.8. $\rho(\varphi) = a\sin 3\varphi, a > 0$. **10.9.** $\rho(\varphi) = a(1 + \cos\varphi), a > 0$.

III. Find all asymptotes of curves:

10.10. $f(x) = \dfrac{x^4}{(1+x)^3}$.

10.11. $f(x) = x + \operatorname{arctg} x$.

10.12. $f(x) = (x+2)e^{\frac{1}{x}}$.

10.13. $f(x) = 2 + \ln\dfrac{x+1}{x-2}$.

10.14. $(x + y + 1)^2 = x^2 + 1$.

10.15. $y^2(x^2+1) = x^2(x^2-1)$.

10.16. $\begin{cases} x(t) = \dfrac{2e^t}{t-1}; \\ y(t) = \dfrac{te^t}{t-1}. \end{cases}$

10.17. $\begin{cases} x(t) = \dfrac{3at}{1+t^3}; \\ y(t) = \dfrac{3at^2}{1+t^3}. \end{cases}$

IV. Explore and plot graphs of given function $y = f(x)$; in tasks 10.24 and 10.25 switch to the polar coordinate system.

10.18. $y = x^2e^{-2x}$.

10.19. $y = \dfrac{x^3+4}{x^2}$.

10.20. $y = \sqrt[3]{x^3 - 2x^2}$.

10.21. $y = \dfrac{x^3}{1-x^2}$.

10.22. $y = \dfrac{\ln x}{\sqrt{x}}$.

10.23. $y = (1+x)^{\frac{1}{x}}$.

10.24. $(x^2 + y^2)^3 = 4a^2x^2y^2$.

10.25. $x^4 + y^4 = a^2(x^2 + y^2)$.

10.26. $\begin{cases} x(t) = te^t; \\ y(t) = te^{-t}. \end{cases}$

10.27. $\begin{cases} x(t) = \dfrac{3t}{1+t^3}; \\ y(t) = \dfrac{3t^2}{1+t^3}. \end{cases}$

Answers and instructionsy

10.1. $(1;3)$ – convex; $(-\infty;1)$, $(3;+\infty)$ – concave; $M_1(1;6)$, $M_2(3;10)$.

10.2. $(-\infty;1)$ – convex; $(1;+\infty)$ – concave; $M(1;e^{-2})$.

10.3. $(-2\sqrt{3};2\sqrt{3})$ – convex; $(-\infty;-2\sqrt{3})$, $(2\sqrt{3};+\infty)$ – concave; $M_{1,2}(\pm2\sqrt{3};4)$.

10.4. $(-1;3)$ – convex; $(-\infty;-1)$, $(3;+\infty)$ – concave; $M_1(-1;-12)$, $M_2(3;-184)$.

10.5. $(1,4)$, $(1,-4)$.

10.6. $t_n = -\frac{\pi}{4} + \pi n$, $n \in Z$ **.**

10.7. $t_{1,2} = \pm\frac{\pi}{3} + 2\pi n$, $t_3 = \pi n$.

10.8. $\varphi_{1,2} = \pm \operatorname{arctg}\sqrt{6+\sqrt{33}} + \pi n$, $\varphi_{3,4} = \pm \operatorname{arctg}\sqrt{6-\sqrt{33}} + \pi n$, $n \in Z$.

10.9. $\varphi_1 = \frac{2\pi}{3} + 2\pi n$, $\varphi_2 = \frac{4\pi}{3} + 2\pi n$, $\varphi_3 = \pi n$, $n \in Z$.

10.10. $x = -1,\ y = x - 3$.

10.11. $y = x - \pi/2$ at $x \to -\infty$, $y = x + \pi/2$ at $x \to +\infty$.

10.12. $x = 0,\ y = x + 3$.

10.13. $x = -1,\ x = 2,\ y = 2$.**10.14.** $y = -1$, $y = -2x$. **10.15.** $y = \pm x$.

10.16. $y = \frac{1}{2}x + e$.

10.17. $y + x + a = 0$

10.22. function scope: $x > 0$; zero function $x = 1$; $y_{max}(e^2) = 2/e$; $M\ (e^{8/3}; 8/3\, e^{-4/3})$ – inflection point; asymptotes: $x = 0$ at $x \to +0$, $y = 0$ at $x \to +\infty$.

10.23. function scope: $x > -1,\ x \neq 0$; $x = 0$ – point of repairable gap; there is no extremum, the function is decreasing, concave; there is no extremum, the function is decreasing, concave; asymptotes: $x = -1$ at $x \to -1+0$, $y = 1$ at $x \to +\infty$.

10.24. Four-petal rose. The origin of the coordinates is a double point of self-touch.

10.25. Symmetrical with respect to four axes: $x = 0$, $y = 0$, $y = x$, $y = -x$; a closed line with four turning points: $(a,0)$, $(-a,0)$, $(0,-a)$, $(0,a)$. The origin of the coordinates is an isolated point.

Section III. Integral calculus of FOV
Topic I. The technique of integrating the indefinite integral

PRACTICAL LESSON 11
Table of integrals and differentials. Basic methods of integration

Mastering the technique of integrating an indefinite integral begins with studying the table of integrals. Some of the formulas in this table follow directly from the definition of integration as the inverse of the differentiation operation, the derivative table, and the invariance property (Property 6.6, [1], P. 221). The validity of other formulas can be checked by differentiation.

Let a be an arbitrary function $u = u(x)$ that has a continuous derivative $u'(x)$ on the interval $(a;b)$. Then the following formulas hold for this interval.

11.1. Table of basic integrals

1. $\int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1} + C$, $\alpha \neq -1$, $u = u(x)$.

2. $\int \frac{du}{u} = \ln|u| + C$.

3. $\int a^{u} du = \frac{a^{u}}{\ln a} + C$; $a \neq 0$, $a \neq 1$; $\int e^{u} du = e^{u} + C$.

4. $\int \sin u du = -\cos u + C$.

5. $\int \cos u du = \sin u + C$.

6. $\int \operatorname{sh} u du = \operatorname{ch} u + C$.

7. $\int \operatorname{ch} u du = \operatorname{sh} u + C$.

8. $\int \operatorname{tg} u du = -\ln|\cos u| + C$.

9. $\int \operatorname{ctg} u du = \ln|\sin u| + C$.

10. $\int \frac{du}{\sin^2 u} = -\operatorname{ctg} u + C$.

11. $\int \frac{du}{\cos^2 u} = \operatorname{tg} u + C$.

12. $\int \frac{du}{\operatorname{sh}^2 u} = -\operatorname{cth} u + C$.

13. $\int \frac{du}{\operatorname{ch}^2 u} = \operatorname{th} u + C$.

14. $\int \frac{du}{\sin u} = \ln\left|\operatorname{tg}\frac{u}{2}\right| + C.$

15. $\int \frac{du}{\cos u} = \ln\left|\operatorname{tg}\left(\frac{u}{2} + \frac{\pi}{4}\right)\right| + C.$

16. $\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + C, \; a = const.$

17. $\int \frac{du}{\sqrt{u^2 + A}} = \ln\left|u + \sqrt{u^2 + A}\right| + C, \; a = const.$

18. $\int \frac{du}{u^2 + a^2} = \frac{1}{a}\operatorname{arctg}\frac{u}{a} + C.$

19. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left|\frac{u - a}{u + a}\right| + C.$

20. $\int \sqrt{a^2 - u^2}\,du = \frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\arcsin\frac{u}{a} + C.$

21. $\int \sqrt{u^2 + a^2}\,du = \frac{u}{2}\sqrt{u^2 + a^2} + \frac{a^2}{2}\ln\left|u + \sqrt{u^2 + a^2}\right| + C.$

Reference. We provide formulas for calculating hyperbolic functions:

$$\operatorname{sh} u = \frac{e^u - e^{-u}}{2}; \; \operatorname{ch} u = \frac{e^u + e^{-u}}{2}; \; \operatorname{th} u = \frac{\operatorname{sh} u}{\operatorname{ch} u}; \; \operatorname{cth} u = \frac{\operatorname{ch} u}{\operatorname{sh} u}.$$

Indefinite integrals can be found using ***tabular integration*** and ***several integration methods***. Tabular integration is based on the use of the table of integrals and properties of the integral, particularly ***Property 6.6*** ([1], P. 221**)** in the case when $u = kx, \; u = x + b, \; u = kx + b$ (partial cases). In other cases, various integration methods are used, among which there are basic methods (see below in the text).

Examples 11.1. Calculate indefinite integrals:

a) $\int (5x + 2)^3\,dx$; b) $\int \cos 6x\,dx$; c) $\int \frac{dx}{\sin^2 9x}$; d) $\int e^{7x+3}dx$; e) $\int \frac{dx}{8x + 9}$;

f) $\int \frac{dx}{(x + 2)^2 + 4}$; g) $\int \frac{dx}{\sqrt{9 - (5x + 1)^2}}$; h) $\int \left(2x^6 - \operatorname{tg} 7x + 3^{9x}\right)dx$.

The solution.

a) $\int (5x + 2)^3\,dx = \left|\begin{matrix} u = 5x + 2; \; partial \; case \\ k = 5, \; formula \; №1, \alpha = 3 \end{matrix}\right| = \frac{1}{4}\frac{(5x + 2)^4}{5} + C.$

b) $\int \cos 6x\,dx = \left|\begin{matrix} u = 6x; k = 6, \\ formula \; №5 \end{matrix}\right| = \frac{1}{6}\sin 6x + C.$

c) $\int \frac{dx}{\sin^2 9x} = \left|\begin{matrix} u = 9x; k = 9, \\ formula \; №10 \end{matrix}\right| = -\frac{1}{9}\operatorname{ctg} 9x + C.$

d) $\int e^{7x+3}dx = \left|\begin{matrix} u = 7x+3; k = 7, \\ formula\ №3 \end{matrix}\right| = \frac{1}{7}e^{7x+3} + C$.

e) $\int \frac{dx}{8x+3} = \left|\begin{matrix} u = 8x+3; k = 8, \\ formula\ №2 \end{matrix}\right| = \frac{1}{8}\ln|8x+3| + C$.

f) $\int \frac{dx}{(x+2)^2+4} = \left|\begin{matrix} u = x+2; a = 2, \\ formula\ №18 \end{matrix}\right| = \frac{1}{2}\operatorname{arctg}\left(\frac{x+2}{2}\right) + C$.

g) $\int \frac{dx}{\sqrt{9-(5x+1)^2}} = \left|\begin{matrix} u = 5x+1; a = 3, \\ k = 9,\ formula\ №16 \end{matrix}\right| = \frac{1}{5}\arcsin\left(\frac{5x+1}{3}\right) + C$.

h) We apply *Properties 6.4, 6.5, 6.6* ([1], P. 221) and formulas 1, 3 and 8 of table of basic integrals

$$\int\left(2x^6 - \operatorname{tg}7x + 3^{9x}\right)dx = 2\int x^6dx - \int\operatorname{tg}7x\,dx + \int 3^{9x}dx =$$

$$= \frac{2x^7}{7} + \frac{1}{7}\ln|\cos 7x| + \frac{3^{9x-2}}{\ln 3} + C .$$

11.2. Basic methods of integrating indefinite integrals: method of direct integration

Calculating integrals with the help of the main Properties (*Properties 6.1 – 6.6*, ([1], P. 221)) of the indefinite integral, algebraic transformations of integral expressions and the table of integrals is called ***direct integration***.

Example 11.2. Find $\int\left(3x^2 - \frac{1}{\cos^2 5x} + \frac{4}{1+(2x+5)^2}\right)dx$.

The solution.

$$I = \int\left(3x^2 - \frac{1}{\cos^2 5x} + \frac{4}{1+(2x+5)^2}\right)dx =$$

$$= \int 3x^2dx - \int\frac{dx}{\cos^2 5x} + \int\frac{4dx}{1+(2x+5)^2} = I_1 + I_2 + I_3 ,$$

$$I_1 = \int 3x^2dx = |\text{by property №4}| = 3\int x^2dx = x^3 + C_1 ,$$

$$I_2 = -\int\frac{dx}{\cos^2 5x} = |formula\ №11| = -\frac{1}{5}\operatorname{tg}5x + C_2 ,$$

$$I_3 = \int\frac{4dx}{1+(2x+5)^2} = \left|\begin{matrix} formula\ №18; \\ k = 2 \end{matrix}\right| = 2\operatorname{arctg}(2x+5) + C_3 .$$

Finally we have

$$I = x^3 - \frac{1}{5}\operatorname{tg} 5x + 2\operatorname{arctg}\left(2x+5\right) + C, \left(C = C_1 + C_2 + C_3\right).$$

Example 11.3. Find the integral $\int\left(\cos^{-2} x \cdot \sin^{-2} x + \sin^2 x\right)dx$.

The solution. When integrating trigonometric expressions, identities are used:

$$\sin^2 x + \cos^2 x = 1; \quad \sin^2 x = (1-\cos 2x)/2; \quad \cos^2 x = (1+\cos 2x)/2;$$

$$\operatorname{tg}^2 x = \sec^2 x - 1; \quad \operatorname{ctg}^2 x = \operatorname{cosec}^2 x - 1, \text{ etc.;}$$

$$I = \int\left(\frac{1}{\cos^2 x \sin^2 x} + \sin^2 x\right)dx = \left|\text{by property №5}\right| = \int\frac{dx}{\cos^2 x \sin^2 x} +$$

$$+\int \sin^2 x\,dx = \int\frac{\cos^2 x + \sin^2 x}{\cos^2 x \cdot \sin^2 x}dx + \int\frac{1-\cos 2x}{2}dx =$$

$$= \int\frac{dx}{\sin^2 x} + \int\frac{dx}{\cos^2 x} + \frac{1}{2}\int dx - \frac{1}{2}\int\cos 2x\,dx = -\operatorname{ctg} x + \operatorname{tg} x + \frac{1}{2}x -$$

$$-\frac{1}{2}\cdot\frac{1}{2}\sin 2x + C = -2\operatorname{ctg} 2x + \frac{x}{2} - \frac{1}{4}\sin 2x + C.$$

Example 11.4. Find the integral $\int\frac{1+2x^2}{x^2+x^4}dx$.

The solution.

$$I = \int\frac{1+2x^2}{x^2+x^4}dx = \int\frac{1+2x^2}{x^2\left(x^2+1\right)}dx = \left|\begin{array}{l} 1+2x^2 = 1+x^2+x^2 = \\ = \left(1+x^2\right)+x^2, \quad \dfrac{a+b}{ab} = \\ = \dfrac{a}{ab} + \dfrac{b}{ab} = \dfrac{1}{b} + \dfrac{1}{a} \end{array}\right| =$$

$$= \int\frac{\left(1+x^2\right)+x^2}{x^2\left(1+x^2\right)}dx = \int\frac{\left(1+x^2\right)dx}{x^2\left(1+x^2\right)} + \int\frac{x^2dx}{x^2\left(1+x^2\right)} = \int\frac{dx}{x^2} + \int\frac{dx}{1+x^2} =$$

$$= \left|\begin{array}{l} formula\ №1; \alpha = -2; \\ formula\ №18; a = 1 \end{array}\right| = -\frac{1}{x} + \operatorname{arctg} x + C.$$

Example 11.5. Find the integral $\int 2^{-x}\left(3\cdot 2^x - 2\cdot 3^x\right)dx$.

The solution.

$$I = \int\frac{3\cdot 2^x - 2\cdot 3^x}{2^x}dx = \int\left(\frac{3\cdot 2^x}{2^x} - \frac{2\cdot 3^x}{2^x}\right)dx =$$

$$= \int\left(3 - 2\cdot\frac{3^x}{2^x}\right)dx = \left|\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x\right| = \int 3dx - \int 2\left(\frac{3}{2}\right)^x dx = 3\int dx - 2\int\left(\frac{3}{2}\right)^x dx =$$

$$= 3x - 2\frac{\left(3/2\right)^x}{\ln 3/2} + C = \left|\int a^u du = \frac{a^u}{\ln a} + C\right| = 3x - \frac{2}{\ln 3/2}\cdot\left(3/2\right)^x + C\,.$$

11.3. The variable replacement method: entering the function under the sign of the differential

Before studying this material, you need to repeat the table of derivatives. The specified method is based on the ***invariance property of the indefinite integral*** and on the definition of the first differential of the function.

Recall that the first differential of an arbitrary differentiable function $y = \varphi(x)$ is found using the formula $dy = d\varphi(x) = \varphi'(x)dx$.

Let the sought integral have the form $\int g(x)dx$. If for the integral function $y = g(x)$ it is possible to choose such a differentiating function such that $g(x) = f(u(x))u'(x)$, and the function $f(u)$, in turn, has a simple (for example, tabular) original $F(u)$, then the given integral takes the form

$$\int g(x)dx = \int f(u(x))u'(x)dx \qquad (11.1)$$

and, taking into account the introduction of the function $u'(x)$ under the sign of the differential, is integrated as follows:

$$\int g(x)dx = \int f(u)u'(x)dx = \left|u'(x)dx = du(x)\right| = \int f(u)du = F(u) + C\,.$$

When applying this method, the following ***formulas for differentials***, which are summarized in a ***table***, are most often used. They need to be understood and remembered. Therefore, let $u = u(x)$ is a continuously differentiable function on the interval (a,b). Then the following relations for differentials hold (***table of differentials***):

1. $u^\alpha du = \dfrac{1}{\alpha + 1}d\left(u^{\alpha+1}\right)$.

2. $\dfrac{du}{u^2} = -d\left(\dfrac{1}{u}\right)$.

3. $\dfrac{du}{\sqrt[n]{u}} = \dfrac{n}{n-1}d\left(\sqrt[n]{u^{n-1}}\right)$.

4. $\sqrt[n]{u}\,du = \frac{n}{n+1} d\left(\sqrt[n]{u^{n+1}}\right).$

5. $e^u du = d(e^u)$.

6. $a^u du = \frac{1}{\ln a} d(a^u)$.

7. $\frac{du}{u} = d(\ln u)$.

8. $\sin u\, du = -d(\cos u)$.

9. $\cos u\, du = d(\sin u)$.

10. $\frac{du}{\cos^2 u} = d(\operatorname{tg} u)$.

11. $\frac{du}{\sin^2 u} = -d(\operatorname{ctg} u)$.

12. $\frac{du}{\cos u} = d\left(\ln\left|\operatorname{tg}\frac{u}{2}\right|\right).$

13. $\frac{du}{\sin u} = d\left[\ln\left|\operatorname{tg}\left(\frac{u}{2}+\frac{\pi}{4}\right)\right|\right].$

14. $\frac{du}{\sqrt{a^2-u^2}} = d(\arcsin\frac{u}{a}) = -d\left(\arccos\frac{u}{a}\right).$

15. $\frac{du}{a^2+u^2} = \frac{1}{a} d\left(\operatorname{arctg}\frac{u}{a}\right) = -\frac{1}{a} d\left(\operatorname{arcctg}\frac{u}{a}\right).$

16. $\frac{du}{\sqrt{u^2+A}} = d\ln\left|u+\sqrt{u^2+A}\right|.$

17. $\frac{du}{u^2-a^2} = \frac{1}{2a} d\ln\left|\frac{u-a}{u+a}\right|.$

18. Based on the properties of the differential, we have:

a) $df(x) = d(f(x)+a)$; b) $d(kf(x)) = kdf(x)$; where $a, k -$ arbitrary nubers.

Example 11.6. Find the integral $\int\frac{xdx}{x^2+3}$.

The solution.

$$\int\frac{xdx}{x^2+3} = \left|\begin{matrix} formula\ №7; u = x^2+3; \alpha = 2 \\ d(x^2+3) = 2xdx \end{matrix}\right| =$$

$$= \frac{1}{2}\int\frac{2xdx}{x^2+3} = \frac{1}{2}\int\frac{d\left(x^2+3\right)}{x^2+3} = \left|u = x^2+3\right| = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2+3\right| + C .$$

Example 11.7. Find the integral $\int x^3\sqrt{2-3x^4}dx$.

The solution.

$$\int x^3\sqrt{2-3x^4}dx = \left| \begin{matrix} formula\ №4; n=2; u=2-3x^4 \\ d(2-3x^4) = -12x^3dx \end{matrix} \right| =$$

$$= -\frac{1}{12}\int\sqrt{2-3x^4}\left(-12\right)x^3dx = -\frac{1}{12}\int\sqrt{2-3x^4}d\left(2-3x^4\right) = -\frac{1}{12}\int\sqrt{u}du =$$

$$= -\frac{1}{12}\int u^{\frac{1}{2}}du = -\frac{1}{12}\frac{u^{\frac{3}{2}}}{3/2} + C = -\frac{2}{12\cdot 3}\sqrt{u^3} + C = -\frac{1}{18}\sqrt{\left(2-3x^4\right)^3} + C\ .$$

Example 11.8. Find the integral $\int\frac{x^3dx}{\sqrt{1-x^8}}$.

The solution.

$$\int\frac{x^3dx}{\sqrt{1-x^8}} = \int\frac{x^3dx}{\sqrt{1-(x^4)^2}} = \left| \begin{matrix} formula\ №14; u=x^4 \\ du = d(x^4) = 4x^3dx; x^3dx = \frac{du}{4} \end{matrix} \right| =$$

$$= \frac{1}{4}\int\frac{4x^3dx}{\sqrt{1-\left(x^4\right)^2}} = \frac{1}{4}.\int\frac{d\left(x^4\right)}{\sqrt{1-\left(x^4\right)^2}} = \frac{1}{4}\int\frac{du}{\sqrt{1-u^2}} = = \frac{1}{4}\arcsin u + C = \frac{1}{4}\arcsin\left(x^4\right) + C.$$

Example 11.9. Find the integral $\int\frac{\cos xdx}{\sin^5 x}$.

The solution.

$$\int\frac{\cos xdx}{\sin^5 x} = \left| \begin{matrix} formula\ №1; u=\sin x; \\ d(\sin x) = \cos xdx; \alpha = -5 \end{matrix} \right| = \int\frac{d(\sin x)}{\sin^5 x} = \int\frac{du}{u^5} =$$

$$= \int u^{-5}du = \frac{u^{-4}}{-4} + C = -\frac{1}{4\sin^4 x} + C\ .$$

Example 11.10. Find the integral $\int\frac{x+\arcsin^2 2x}{\sqrt{1-4x^2}}dx$.

The solution.

$$I = \int\frac{x+\arcsin^2 2x}{\sqrt{1-4x^2}}dx = \int\frac{xdx}{\sqrt{1-4x^2}} + \int\frac{\arcsin^2 2x}{\sqrt{1-4x^2}}dx = I_1 + I_2\ ;$$

$$I_1 = \int\frac{xdx}{\sqrt{1-4x^2}} = \left| \begin{matrix} formula\ №3; u=1-4x^2; n=2 \\ du = d(1-4x^2) = -8xdx \end{matrix} \right| =$$

$$= -\frac{1}{8}\int\frac{-8xdx}{\sqrt{1-4x^2}} = -\frac{1}{8}\int\frac{d\left(1-4x^2\right)}{\sqrt{1-4x^2}} = -\frac{1}{8}\int\frac{du}{\sqrt{u}} = -\frac{1}{8}\cdot 2\sqrt{u}+C = -\frac{1}{4}\sqrt{1-4x^2}+C_1;$$

$$I_2 = \int\frac{\arcsin^2 2xdx}{\sqrt{1-4x^2}} = \left|\begin{array}{l} formulas\ №1\ and\ 14; u=\arcsin 2x; \\ du = d(\arcsin 2x) = \dfrac{2dx}{\sqrt{1-4x^2}}; \alpha = 2 \end{array}\right| =$$

$$= \frac{1}{2}\int\arcsin^2 2x\frac{2dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int\arcsin^2 2xd(\arcsin 2x) = \frac{1}{2}\int u^2du = \frac{u^3}{6}+C_2 = \frac{\arcsin^3 2x}{6}+C_2.$$

Finally we have:

$$I = I_1 + I_2 = -\frac{1}{4}\sqrt{1-4x^2} + \frac{\arcsin^3 2x}{6} + C, \quad \left(C = C_1 + C_2\right).$$

11.4. The method of replacing the variable: subtracting the function from the sign of the differential

The essence of this method is to introduce a ***new*** integration ***variable*** instead x.

Theorem. *Let* $F(x)$ *is the primitive function* $f(x)$ *on the interval* P*, i.e.:*

$$\int f(x)dx = F(x) + C,$$

and let the function $x=\varphi(t)$ *be defined and differentiable on the interval* P_1*, and the set of values* $\varphi(P_1)$ *of this function is contained in the set* P*, i.e.* $\varphi(P_1)\subset P$*. Then the variable replacement formula is valid*

$$\int f\left(\varphi(t)\right)\cdot\varphi'(t)dt = F\left(\varphi(t)\right) + C. \qquad (11.2)$$

Remark 11.1. After finding the indefinite integral by the variable replacement method, it is ***necessary*** to move from the ***new variable*** t (or u) to the ***old variable*** x.

Remark 11.2. Change of variable is not applicable for every integral. It should be noted that the choice of the correct replacement of the variable, which leads to the goal most quickly, is largely determined by the experience of the researcher.

Example 11.11. Find the integral $\int\frac{xdx}{\sqrt{x-3}+5}$.

The solution. It is often necessary to introduce a new variable instead of roots. Let's replace $x=t^2+3$ with a new variable t, as a result we get:

$$\int \frac{x\,dx}{\sqrt{x-3}+5} = \left| \begin{matrix} x = t^2+3, & dx = 2t\,dt, \\ x-3 = t^2, & t = \sqrt{x-3}. \end{matrix} \right| = 2\int \frac{\left(t^3+3t\right)dt}{t+5} =$$

$$= \left| \frac{t^3+3t}{t+5} = t^2 - 5t + 28 - \frac{140}{t+5} \right| = 2\int \left(t^2 - 5t + 28 - \frac{140}{t+5} \right) dt =$$

$$= 2\int t^2 dt - 10\int t\,dt + 56\int dt - 280\int \frac{dt}{t+5} =$$

$$= 2\frac{t^3}{3} - 10\frac{t^2}{2} + 56t - 280\ln|t+5| + C = \frac{2}{3}\sqrt{(x-3)^3} - 5(x-3) +$$

$$+56\sqrt{x-3} - 280\ln\left|\sqrt{x-3}+5\right| + C.$$

11.5. Method of integration by parts

Let $u = u(x)$ and $v = v(x)$ be continuously differentiable functions on some interval. The formula for integration by parts has the form

$$\int u dv = uv - \int v du . \qquad (11.3)$$

Formula (11.3) shows that the calculation of the integral $\int u dv$ is reduced to the calculation of the integral $\int v du$, which can be simpler than the given one, or even tabular. The resulting formula is used in those cases when the integrand expression $f(x)dx$ can be represented in the form udv. At the same time, it should be borne in mind that the function $u(x)$ should include factors that are simplified during differentiation. As a rule, the integral expression, which is the product udv, can be factored u and dv in several ways. However, the integral function through u and dv must be represented so that the integral $\int v du$ is simpler than the integral $\int u dv$.

Let's consider the ***main types of integrals***, which are convenient to find by the method of integration by parts.

1. If the integral expression is the product of an exponential or trigonometric function by a polynomial, then the polynomial must be taken as the function u, and the remaining expression as dv. These are integrals of the form:

$$\int P_n(x) \cdot e^{kx} dx; \quad \int P_n(x) \sin kx dx; \quad \int P_n(x) \cos kx dx ,$$

where $P_n(x)$ is a polynomial of degree n, and k is a real number.

2. If the integral expression contains the product of a logarithmic or inverse trigonometric function by a polynomial, that is, the integrals have the form:

$$\int P_n(x) \ln x dx; \quad \int P_n(x) \arcsin x dx; \quad \int P_n(x) \arccos x dx; \quad \int P_n(x) \operatorname{arctg} x dx ,$$

then the function u should be taken as the logarithmic function, or the inverse of the trigonometric function, and as dv the expression $P_n(x)dx$, where $P_n(x)$ is a polynomial, i.e. $dv = P_n(x)dx$.

3. Sometimes the formula for integration by parts has to be applied ***several times***. To find integrals of the form:

$$\int P_n(x)e^{kx}dx\,,\ \int P_n(x)\sin kx dx\,,\ \int P_n(x)\cos kx dx\,,$$

where $P_n(x)$ is a polynomial of degree n, it is necessary to apply the formula for integration by parts as many times as the degree of the polynomial. At the same time, the function $u(x)$ is taken as a power function each time.

4. In some cases, repeated application of the formula for integration by parts leads to linear equations relative to the desired integral. Solving this equation gives us the required integral. Such integrals include the following:

$$\int e^{mx}\sin nx dx\,,\ \int e^{mx}\cos nx dx\,,\ \int \sin(\ln x)dx\,,\ \int \cos(\ln x)dx\,,$$

where m, n – are real numbers.

Remark. Note that when finding the function v by differential dv, it is assumed that the constant $C = 0$, since this constant does not affect the final result. Indeed, by substituting $v + C$ into the formula for integration by parts, we obtain

$$\int u d(v+C) = u(v+C) - \int (v+C)du \Rightarrow \int u dv = uv + Cu - \int v du - Cu \Rightarrow$$

$$\Rightarrow \int u dv = uv - \int v du.$$

Let's consider a number of examples.

Examples 11.12.

1. $\int (2x+1)\sin x dx = \begin{vmatrix} u = 2x+1; & du = 2dx \\ dv = \sin x dx; & v = \int \sin x dx = -\cos x \end{vmatrix} =$

$$= -(2x+1)\cos x - \int(-\cos x)2dx = -(2x+1)\cos x + 2\sin x + C\,.$$

2. $\int x^3 \ln x dx = \begin{vmatrix} u = \ln x; \ du = \dfrac{dx}{x} \\ dv = x^3dx; v = \int x^3 dx = \dfrac{x^4}{4} \end{vmatrix} = \dfrac{1}{4}x^4\ln x - \dfrac{1}{4}\int x^4\dfrac{dx}{x} = \dfrac{1}{4}\ln x - \dfrac{x^4}{16} + C\,.$

3. $\int \arcsin x dx = \begin{vmatrix} u = \arcsin x; & du = \dfrac{1}{\sqrt{1-x^2}} \\ dv = dx; & v = \int dx = x \end{vmatrix} =$

$$= x\cdot\arcsin x - \int\frac{xdx}{\sqrt{1-x^2}} = x\cdot\arcsin x + \frac{1}{2}\int\frac{d(1-x^2)}{\left(1-x^2\right)^{\frac{1}{2}}} = x\cdot\arcsin x + \sqrt{1-x^2} + C\,.$$

4. $I=\int e^x\sin 2x\,dx=\left|\begin{matrix} e^x=u; & du=e^x dx \\ \sin 2x\,dx=dv; & v=-\frac{1}{2}\cos 2x\end{matrix}\right|=$

$$=-\frac{1}{2}e^x\cos 2x+\frac{1}{2}\int e^x\cos 2x\,dx=\left|\text{let's re-apply integration by parts}\right|=$$

$$=\left|\begin{matrix} u=e^x; & du=e^x dx \\ dv=\cos 2x\,dx; & v=\frac{1}{2}\sin 2x\end{matrix}\right|=-\frac{1}{2}e^x\cos 2x+\frac{1}{4}e^x\sin 2x-\frac{1}{4}\int e^x\sin 2x\,dx.$$

We have: $\int e^x\sin 2x\,dx=-\frac{1}{2}e^x\cos 2x+\frac{1}{4}e^x\sin 2x-\frac{1}{4}\int e^x\sin 2x\,dx$,

or $I=-\frac{1}{2}e^x\cos 2x+\frac{1}{4}e^x\sin 2x-\frac{1}{4}I$.

We have obtained an equation from which we determine the required integral I:

$$I+\frac{1}{4}I=\frac{1}{4}(\sin 2x-2\cos 2x)\cdot e^x;\quad \frac{5}{4}I=\frac{1}{4}(\sin 2x-2\cos 2x)e^x.$$

Finally, we get: $\int e^x\sin 2x\,dx=\frac{1}{5}(\sin 2x-2\cos 2x)e^x+C$.

Tasks for classroom and independent work

I. Find indefinite integrals by direct integration or change of variable method:

11.1. $\int\frac{2^{x+1}-5^{x-1}}{10^x}dx$.

11.2. $\int\frac{dx}{a^2x^2-b^2}$.

11.3. $\int\frac{dx}{(1+x^2)\cdot(\operatorname{arctg}x)^{5/6}}$.

11.4. $\int\frac{dx}{e^x+e^{-x}}$.

11.5. $\int\frac{dx}{x\ln x\ln(\ln x)}$.

11.6. $\int\frac{dx}{(1+x)\sqrt{x}}$.

11.7. $\int\frac{dx}{\sin^2 x\cdot\sqrt[4]{ctgx}}$.

11.8. $\int\frac{dx}{\sqrt{1-x^2}\cdot\arcsin^2 x}$.

11.9. $\int\frac{(x+3)dx}{\sqrt{3+4x-4x^2}}$.

11.10. $\int e^{4\sin x}\cdot\cos x\cdot dx$.

11.11. $\int\frac{x^2+1}{x^4+1}dx$.

11.12. $\int\frac{\ln x\,dx}{x\sqrt{1+\ln x}}$.

11.13. $\int \frac{\sin x dx}{\sqrt[6]{\cos^5 x}}$.

11.14. $\int \sin 8x \cos 6x dx$.

11.15. $\int \frac{x^2 dx}{\sqrt{x-5}+2}$.

11.16. $.\int \frac{dx}{\text{ch}\, x}$

11.17. $\int \frac{(x-3)dx}{x^2+6x+10}$.

11.18. $\int \frac{x^2 dx}{\sqrt{2-x}}$.

11.19. $\int \frac{arctg\sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x}$.

11.20. $\int \frac{(x+1)dx}{\sqrt{x^2+x+1}}$.

11.21. $\int \frac{dx}{e^{2x}+8e^x+17}$.

11.22. $\int \frac{(e^x+1)dx}{(e^x-1)^2}$.

II. Find indefinite integrals by integration by parts:

11.23. $\int (x+1)^2 e^{4x} dx$.

11.24. $\int \arccos 3x dx$.

11.25. $\int x^{5/3} \ln x dx$.

11.26. $\int \text{arctg}\, 2x dx$.

11.27. $\int e^{2x} \cos 3x dx$.

11.28. $\int \sin(\ln x) dx$.

11.29. $\int (x^2+2x+3)\cos 2x dx$.

11.30. $\int x^5 \,\text{arctg}\, x dx$.

11.31. $\int \frac{x \arcsin 2x}{\sqrt{1-4x^2}} dx$.

11.32. $\int \frac{x dx}{\cos^2 x}$.

11.33. $\int x^4 \ln(x^2-1) dx$.

11.34. $\int e^{3x} \ln(e^x+4) dx$.

Answers and instructions

11.1. $-\frac{2\cdot 5^{-x}}{\ln 5}+\frac{2^{-x}}{5\ln 2}+C$.

11.2. $\frac{1}{2ab}\ln\frac{ax-b}{ax+b}+C$.

11.3. $6(\text{arctg}\, x)^{1/6}+C$.

11.4. $\text{arctg}\, e^x + C$.

11.5. $\ln\left|\ln(\ln x)\right|+C$.

11.6. $2\,\text{arctg}\sqrt{x}+C$.

11.7. $-\frac{4}{3}\sqrt[4]{ctg^3 x}+C$.

11.8. $-\frac{1}{\arcsin x}+C$.

11.9. $-\frac{1}{4}\sqrt{3+4x-4x^2}+\frac{7}{4}\arcsin\left(x-\frac{1}{2}\right)+C$.

11.10. $\frac{1}{4}e^{4\sin x}+C$.

11.11. $\frac{1}{\sqrt{2}}\text{arctg}\frac{x^2-1}{x\sqrt{2}}+C$.

11.12. $\frac{2}{3}(\ln x-2)\sqrt{1+\ln x}+C$. **11.13.** $-6\sqrt[6]{\cos x}+C$.

11.14. $-\frac{1}{4}(\cos 2x+\frac{1}{7}\cos 14x)+C$.

11.15. $\frac{2}{5}t^5-t^4+\frac{28}{3}t^3-28t^2+162t-324\ln|t+2|+C,\ t=\sqrt{x-5}$.

11.16. $2\operatorname{arctg} e^x+C$.

11.17. $\frac{1}{2}\ln(x^2+6x+10)-6\,arctg(x+3)+C$.

11.18. $-\frac{2}{15}(3x^2+8x+32)\sqrt{2-x}+C$. **11.19.** $(\operatorname{arctg}\sqrt{x})^2+C$.

11.20. $\sqrt{x^2+x+1}+\frac{1}{2}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right)+C$.

11.21. $\frac{x-4\operatorname{arctg}(e^x+4)}{17}-\frac{\ln[(e^x+4)^2+1]}{34}+C$.

11.22. $x-\ln|e^x-1|-\frac{2}{e^x-1}+C$.

11.23. $\frac{e^{4x}}{4}\left((x+1)^2-\frac{1}{2}(x+1)+\frac{1}{8}\right)+C$.

11.24. $x\arccos 3x-\frac{1}{3}\sqrt{1-9x^2}+C$. **11.25.** $\frac{3}{8}x^{8/3}(\ln x-3/8)+C$.

11.26. $x\operatorname{arctg} 2x-\frac{1}{4}\ln(1+4x^2)+C$. **11.27.** $\frac{e^{2x}}{13}(3\sin 3x+2\cos 3x)+C$.

11.28. $\frac{x}{2}\big(\sin(\ln x)-\cos(\ln x)\big)+C$.

11.29. $\frac{1}{2}\left((x^2+2x+3)\sin 2x+(x+1)\cos 2x-\frac{1}{2}\sin 2x\right)+C$.

11.30. $xtgx+\ln|\cos x|+C$.

11.31. $\frac{1}{6}(x^6\operatorname{arctg} x-\frac{x^5}{5}+\frac{x^3}{3}-x+\operatorname{arctg} x)+C$.

11.32. $\frac{1}{4}(2x-\sqrt{1-4x^2}\arcsin 2x)+C$.

11.33. $\frac{1}{5}\ln\left|\frac{x+1}{x-1}\right|+\frac{x^5\ln|x^2-1|}{5}-\frac{2x}{5}-\frac{2x^3}{15}-\frac{2x^5}{25}+C$.

11.34. $\frac{2e^{2x}-16e^x+(e^{3x}+64)\ln(e^x+4)}{3}-\frac{e^{3x}}{9}+C$.

PRACTICAL LESSON 12

Proper and improper rational fractions. Technique of integration of rational fractions

First, let's give some definitions.

Definition. *A **fractional rational function** $R(x)$ is a function equal to the ratio of two polynomials:*

$$R(x) = \frac{P_n(x)}{Q_m(x)} = \frac{a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m}, \qquad (12.1)$$

where m, n *– are natural numbers;* b_i, a_j *– are real numbers,* $i = \overline{0, m}$, $j = \overline{0, n}$.

Definition. *If* $n < m$, *then* $R(x)$ *is called a **proper rational fraction**, if* $n < m$, $R(x)$ *is an **improper fraction**.*

Any improper fraction by dividing the numerator by the denominator can be represented as the sum of some polynomial and a proper fraction:

$$\frac{P_n(x)}{Q_m(x)} = W_{n-m}(x) + \frac{R_p(x)}{Q_m(x)}, \qquad (12.2)$$

where $W_{n-m}(x), R_p(x)$ – are polynomials of the appropriate degree, $\dfrac{R_p(x)}{Q_m(x)}$ – is a proper fraction, $p < m$.

For example, $\dfrac{x^4 + 4}{x^2 + 3x - 1}$ – an improper fraction. Dividing its numerator by the denominator (according to the rule of dividing polynomials by a "corner"), we get:

$$\frac{x^4 + 4}{x^2 + 3x - 1} = x^2 - 3x + 10 + \frac{-33x + 14}{x^2 + 3x - 1}.$$

Conclusion. Since any polynomial is easily integrated, the ***integration of rational functions*** is reduced to the ***integration of proper fractions*** (formula (12.2)).

Next, we will consider simple integrals from an improper rational fraction.

Examples 12.1. Find the integrals:

a) $\int \dfrac{x^2 + 12}{x^2 - 3} dx$;

b) $\int \dfrac{x^3 dx}{x + 2}$.

The solution.

a) Let's take the whole part of the integral function. To do this, divide the numerator by the denominator by dividing a polynomial by a polynomial, or add and subtract a number 3 from the numerator and consider the sum of the corresponding fractions. As a result, we have:

$$\frac{x^2-3+3+12}{x^2-3}=\frac{x^2-3}{x^2-3}+\frac{15}{x^2-3}=1+\frac{15}{x^2-3};$$

$$\int\frac{x^2+12}{x^2-3}dx=\int\left(1+\frac{15}{x^2-3}\right)dx=\int dx+15\int\frac{dx}{x^2-3}=x+\frac{15}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|+C.$$

b) In the second example, we divide the numerator by the denominator with a "corner". As a result, we get a polynomial of the second order (as a whole part) and a proper fraction:

$$\int\frac{x^3dx}{x+2}=\left|\begin{array}{ll} \_\; x^3 & \vert\; \underline{x+2\qquad\qquad} \\ \underline{x^3+2x^2} & \quad x^2-2x+4 \\ \_-2x^2 & \\ \quad\underline{-2x^2-4x} & \\ \qquad\_\;4x & \\ \qquad\;\underline{4x+8} & \\ \qquad\quad -8 & \end{array}\right|=\int\left[x^2-2x+4-\frac{8}{x+2}\right]dx=$$

$$=\int x^2dx-2\int xdx+4\int dx-8\int\frac{dx}{x+2}=\frac{x^3}{3}-x^2+4x-8\ln|x+2|+C.$$

12.1. Integration of a proper rational fraction. Decomposition of a proper rational fraction into elementary fractions

Before studying this material, it is advisable to repeat actions on algebraic expressions and factoring polynomials. First, we will give the definition of elementary fractions.

Definition. ***Elementary (simple) fractions*** *are called proper rational fractions of the following four types:*

$$1)\ \frac{A}{x-a};\qquad 2)\ \frac{A}{(x-a)^n},\quad (n=2,3,\cdots);$$

$$3)\ \frac{Mx+N}{x^2+px+q};\qquad 4)\ \frac{Mx+N}{(x^2+px+q)^n},\quad (n\ge 2),\qquad (12.3)$$

where A, a, M, N, p, q *are* ***real*** *numbers, and the trinomial* x^2+px+q ***has no real roots****, i.e.* $p^2-4q<0$.

Now consider a proper rational fraction $\frac{R_p(x)}{Q_m(x)}$, ($p < m$). For example, the fraction $\frac{x^2+1}{x^5+2x+3}$ is ***proper*** because $p = 2 < 5 = m$.

Let's formulate the Basic Theorem on the decomposition of a proper fraction into elementary ones.

Basic Theorem. *If the denominator of* ***a proper rational fraction*** $\frac{R_p(x)}{Q_m(x)}$ *is factored* $\left(m = \alpha + \beta + \cdots + \delta + 2(\mu + \cdots + \nu)\right)$:

$$Q_m(x) = b_0(x-a)^{\alpha} \cdot (x-b)^{\beta} \cdot \cdots \cdot (x-c)^{\delta} \cdot \left(x^2 + px + q\right)^{\mu} \cdot \cdots \cdot \left(x^2 + lx + s\right)^{\nu},$$

then this fraction can be presented as a ***sum of elementary fractions***:

$$\frac{R_p(x)}{Q_m(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_\alpha}{(x-a)^\alpha} + \cdots +$$

$$+\frac{B_1}{(x-b)} + \frac{B_2}{(x-b)^2} + \cdots + \frac{B_\beta}{(x-b)^\beta} + \cdots + \frac{C_1}{(x-c)} + \frac{C_2}{(x-c)^2} + \cdots + \frac{C_\delta}{(x-c)^\delta} +$$

$$+\frac{M_1x + N_1}{x^2 + px + q} + \frac{M_2x + N_2}{\left(x^2 + px + q\right)^2} + \cdots + \frac{M_\mu x + N_\mu}{\left(x^2 + px + q\right)^\mu} + \cdots +$$

$$+\frac{L_1x + S_1}{x^2 + lx + s} + \frac{L_2x + S_2}{\left(x^2 + lx + s\right)^2} + \cdots + \frac{L_\nu x + S_\nu}{\left(x^2 + lx + s\right)^\nu}. \qquad (12.4)$$

Definition. *Expression* (12.4) *is called the* ***decomposition of a proper rational fraction into elementary fractions***.

Numbers $A_1, A_2, \cdots, A_\alpha, \cdots, C_1, C_2, \cdots, C_\delta, M_1, N_1, \cdots, M_\mu, N_\mu, \cdots, L_1, S_1, \cdots,$ L_ν, S_ν are some unknown real numbers (***undefined coefficients***), which can be found using ***two main methods***.

A. ***The method of comparing coefficients with the same powers.*** We will present the algorithm of its application.

1. Multiply both parts of expression (12.4) by $Q_m(x)$. As a result, we get two identically equal polynomials – a known polynomial $R_p(x)$ and a polynomial with unknown (undefined) coefficients $A_1, \cdots, S_\nu$.

2. Equate their coefficients with the same powers x and write $A_1, \cdots, S_\nu$. own the system of linear algebraic equations (SLAE) with respect to the unknown coefficients.

3. Solve the obtained SLAE by finding the unknowns $A_1, \cdots, S_\nu$.

In addition to the method of comparing coefficients, the ***method of individual values of the argument*** is also used.

B. ***The method of individual values of the argument*.** The algorithm of its application is as follows:

1. Multiply both parts of expression (12.4) by $Q_m(x)$. As a result, we get two identically equal polynomials: a known polynomial $R_p(x)$ and a polynomial with unknown coefficients $A_1, \cdots, S_v$.

2. Give the variable x specific values as many times as there are unknown coefficients in the indicated equality. As a result, we will get SLAE relative to the unknown coefficients $A_1, \cdots, S_v$.

3. Solve the obtained SLAE by finding the unknowns $A_1, \cdots, S_v$.

Remark 12.1. SLAE is greatly simplified if the variable x is given the value of the roots of the denominator $Q_m(x)$.

Example 12.2. Express a fraction $\dfrac{-8x+6}{x\cdot(x-1)\cdot(x-3)}$ in terms of elementary fractions.

***The solution*.**

$$\frac{-8x+6}{x\cdot(x-1)\cdot(x-3)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3};$$
$$-8x+6 \equiv A\cdot(x-1)\cdot(x-3) + Bx(x-3) + Cx(x-1).$$

Let's use the second method. If we put in this identity:

If $x=0$, then $6 = 3A \Rightarrow A = 2$,

If $x=1$, then $-2 = -2B \Rightarrow B = 1$,

If $x=3$, then $-18 = 6\cdot C \Rightarrow C = -3$.

So, we have the desired schedule:

$$\frac{-8x+6}{x\cdot(x-1)\cdot(x-3)} = \frac{2}{x} + \frac{1}{x-1} - \frac{3}{x-3}.$$

Remark 12.2. In practical application, the method of individual values of the argument has some advantages over the comparison method. Its implementation leads to the construction and solution of ***sparse*** SLAE with a ***smaller volume of calculations*** compared to the method of uncertain coefficients. However, in well-known manuals and textbooks on higher mathematics , the application of this method is considered only for the case of real roots of the denominator $Q_m(x)$. In this training course, the ***method of individual values of the argument is generalized to the case of complex roots of the denominator***.

The first volume of the textbook "Mathematical Analysis" [1] provides educational material on operations with complex numbers, sufficient for understanding and working out the generalized method of individual values of the argument, including the separation of the imaginary part of the complex number from the real part. As a result of such separation, one complex equation gives rise to two equations with real coefficients.

To illustrate the effectiveness of the generalized method of individual values of the argument for the case of complex roots of the denominator $Q_m(x)$, consider the following example. Recall that the imaginary unit is defined as follows: $i^2+1=0$.

Example 12.3. To represent a fraction $\dfrac{2x^3+3x^2+10x+9}{\left(x^2+1\right)\cdot\left(x^2+2x+5\right)}$ through elementary.

The solution. Let's find the roots of the denominator. All of them are complex numbers: $i, -i, -1+2i, -1-2i$. So, we have a decomposition of the given fraction into the following elementary fractions with unknown coefficients A, B, C, D:

$$\frac{2x^3+3x^2+10x+9}{\left(x^2+1\right)\cdot\left(x^2+2x+5\right)} \equiv \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+2x+5}.$$

Let's multiply both parts of this identity by the original denominator $\left(x^2+1\right)\cdot\left(x^2+2x+5\right)$. After that, we equate the numerator of the given fraction to the numerator with unknown coefficients:

$$2x^3+3x^2+10x+9 \equiv \left(x^2+1\right)\left(Cx+D\right)+\left(x^2+2x+5\right)(Ax+B).$$

Substitute in the identity instead of the value x of the first root $x_1=i$. As a result, we will get two equations with respect to two unknown coefficients A and B:

$$8i+6=(4+2i)(Ai+B) \Rightarrow \begin{cases} 4A+2B=8; \\ 4B-2A=6. \end{cases} \Rightarrow \begin{cases} A=1; \\ B=2. \end{cases}$$

Now we substitute the third root $x_3=2i-1$ of the denominator into the identity:

$$\left[(2i-1)^2+1\right]\left[C(2i-1)+D\right] \equiv 2(2i-1)^3+3(2i-1)^2+10(2i-1)+9.$$

After equivalent transformations in this identity according to the rules of operations on complex numbers, we will again obtain two equations from which we will find the unknown coefficients C and D:

$$-4Di+10C-2D\equiv 4i+12\Rightarrow\begin{cases}D=-1;\\10C-2D=12;\end{cases}\Rightarrow\begin{cases}D=-1;\\C=1.\end{cases}$$

So, we have $\dfrac{2x^3+3x^2+10x+9}{\left(x^2+1\right)\cdot\left(x^2+2x+5\right)}\equiv\dfrac{x+2}{x^2+1}+\dfrac{x-1}{x^2+2x+5}.$

Remark 12.3. Sometimes it is convenient to use a ***combined method***, that is, to determine some of the unknown coefficients by giving the values of the roots of the denominator, and to determine others by the method of comparison.

Example 12.4. Express the fraction $\dfrac{x^2+x+2}{x\cdot(x-1)\cdot\left(x^2+1\right)}$ in terms of elementary fractions.

***The solution*.** Let's represent the given fraction in the form of elementary fractions with undefined coefficients:

$$\frac{x^2+x+2}{x\cdot(x-1)\cdot\left(x^2+1\right)}\equiv\frac{A}{x}+\frac{B}{x-1}+\frac{Cx+D}{x^2+1},$$

or after multiplying by the denominator:

$$x^2+x+2\equiv A\cdot(x-1)\left(x^2+1\right)+Bx\left(x^2+1\right)+x(x-1)(Cx+D)\Rightarrow$$
$$\Rightarrow x^2+x+2\equiv(A+B+C)\cdot x^3+(-A-C+D)\cdot x^2+(A+B-D)x-A.$$

If $x=0$, then $2=-A\Rightarrow A=-2$.

If $x=1$, then $4=2B\Rightarrow B=2$.

Comparing the coefficients at x^3, we get: $0=A+B+C\Rightarrow C=0$. And finally, we find D, by equating the coefficients at x^2:

$$1=-A-C+D\Rightarrow D=-1.$$

So, we finally have

$$\frac{x^2+x+2}{x\cdot(x-1)\cdot\left(x^2+1\right)}=-\frac{2}{x}+\frac{2}{x-1}-\frac{1}{x^2+1}.$$

We get the same result for C and D if we put in the identity $x=i$.

12.2. Integration of elementary fractions

Integrating elementary fractions (12.3) does not cause difficulties, so we will perform their integration without additional explanations. Let us consider the corresponding integrals.

1. $I_1 = \int \frac{A}{x-a}dx = A\int \frac{d(x-a)}{x-a} = A\ln|x-a| + C$.

2. $I_2 = \int \frac{A}{(x-a)^n}dx = A\int (x-a)^{-n}\, d(x-a) = A\frac{(x-a)^{-n+1}}{-n+1} + C$.

3. Consider the integral $\int \frac{Mx+N}{ax^2+bx+c}dx$. This integral reduces to an integral of the form:

$$\int \frac{(Mx+N)dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{(Mx+N)dx}{x^2+\frac{b}{a}x+\frac{c}{a}} = \frac{1}{a}\int \frac{Mx+N}{x^2+px+q}dx = \frac{1}{a}I_3,\ \partial e\ I_3 = \int \frac{Mx+N}{x^2+px+q}dx .$$

$$I_3 = \int \frac{Mx+N}{x^2+2\cdot\frac{p}{2}\cdot x+\frac{p^2}{4}-\frac{p^2}{4}+q}dx = \int \frac{Mx+N}{\left(x+\frac{p}{2}\right)^2+q-\frac{p^2}{4}}dx = \left| \begin{matrix} t = x+\frac{p}{2} \\ dx = dt \end{matrix} \right| =$$

$$= \int \frac{M\left(t-\frac{p}{2}\right)+N}{t^2+\left(q-\frac{p^2}{4}\right)}dt = M\int \frac{t\cdot dt}{t^2+\left(q-\frac{p^2}{4}\right)} + \left(N-M\frac{p}{2}\right)\cdot\int \frac{dt}{t^2+\left(q-\frac{p^2}{4}\right)} = J_1 + J_2.$$

The first of the last two integrals J_1 is calculated as follows:

$$J_1 = \frac{M}{2}\int \frac{2tdt}{t^2+\left(q-\frac{p^2}{4}\right)} = \frac{M}{2}\int \frac{d\left(t^2+q-\frac{p^2}{4}\right)}{t^2+\left(q-\frac{p^2}{4}\right)} = \frac{M}{2}\ln\left|t^2+q-\frac{p^2}{4}\right| + C =$$

$$= \frac{M}{2}\ln\left|x^2+px+q\right| + C.$$

Since $p^2-4q<0$, then the second integral J_2 is calculated according to the table in the form of a function $\mathrm{arctg}\, u(x)$.

Example 12.5. Find the integral: $\int \frac{(4x-3)\,dx}{2x^2-12x+10}$.

The solution.

$$\int \frac{(4x-3)\,dx}{2x^2-12x+10} = \frac{1}{2}\int \frac{(4x-3)\,dx}{x^2-6x+5} = \left| \begin{array}{l} x^2+px+q = \left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4}; \\ x^2-6x+5 = (x-3)^2+5-9 \end{array} \right| =$$

$$= \frac{1}{2}\int \frac{(4x-3)\,dx}{(x-3)^2-4} = \left| \begin{array}{l} x-3=t \\ x=t+3 \\ dx=dt \end{array} \right| = \frac{1}{2}\int \frac{4(t+3)-3}{t^2-4}dt = \frac{1}{2}\int \frac{4t+12-3}{t^2-4}dt =$$

$$= \frac{1}{2}\int \frac{4t+9}{t^2-4}dt = \frac{1}{2}\int \frac{4t}{t^2-4}dt + \frac{9}{2}\int \frac{dt}{t^2-4} = \int \frac{2t}{t^2-4}dt + \frac{9}{2}\int \frac{dt}{t^2-4} =$$

$$= \left|2t\,dt = d\left(t^2-4\right)\right| = \int \frac{d\left(t^2-4\right)}{t^2-4} + \frac{9}{2}\cdot\frac{1}{2\cdot 2}\ln\left|\frac{t-2}{t+2}\right| = \ln\left|t^2-4\right| + \frac{9}{8}\ln\left|\frac{t-2}{t+2}\right| + C =$$

$$= \ln\left|(x-3)^2-4\right| + \frac{9}{8}\ln\left|\frac{x-3-2}{x-3+2}\right| + C = \ln\left|x^2-6x+5\right| + \frac{9}{8}\ln\left|\frac{x-5}{x-1}\right| + C .$$

4. The integral of the form $I_4 = \int \frac{Mx+N}{\left(x^2+px+q\right)^n}dx$, $\left(p^2-4q<0,\ n\ge 2\right)$ reduces to two integrals

$$\int \frac{Mx+N}{\left(x^2+px+q\right)^n}dx = \frac{M}{2}\int \frac{d\left(x^2+px+q\right)-p\,dx}{\left(x^2+px+q\right)^n} + \int \frac{N\,dx}{\left(x^2+px+q\right)^n} =$$

$$= \frac{M}{2}\frac{1}{1-n}\left(x^2+px+q\right)^{1-n} + \left(N-\frac{Mp}{2}\right)\int \frac{dx}{\left(x^2+px+q\right)^n},$$

the first of which is calculated directly, and the second is reduced to the following integral by substitution $t = x+\frac{p}{2}$:

$$I_n = \int \frac{dt}{(t^2+a^2)^n} \quad (n=1,\ 2,\ 3,\ldots) .$$

We will show that there is the following ***recurrent formula*** for calculating such integrals for different natural values n:

$$I_{n+1} = \frac{1}{2na^2}\,\frac{t}{(t^2+a^2)^n} + \frac{2n-1}{2n}\,\frac{1}{a^2}I_n \quad (n=1,\ 2,\ 3,\ldots), \qquad (12.5)$$

where $I_1 = \int \frac{dt}{t^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{t}{a} + C$.

Indeed, integrating the integral I_n by parts, we get:

$$u = \frac{1}{(t^2+a^2)^n}, \qquad dv = dt, \qquad v = t, \qquad du = -\frac{2ntdt}{(t^2+a^2)^{n+1}},$$

$$I_n = \frac{t}{(t^2+a^2)^n} + 2n\int \frac{t^2+a^2-a^2}{(t^2+a^2)^{n+1}}dt =$$

$$= \frac{t}{(t^2+a^2)^n} + 2n\int \frac{dt}{(t^2+a^2)^n} - 2na^2\int \frac{dt}{(t^2+a^2)^{n+1}}; \quad I_n = \frac{t}{(t^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1},$$

or in this way

$$I_{n+1} = \frac{1}{2na^2}\cdot\frac{t}{(t^2+a^2)^n} + \frac{2n-1}{2n}\frac{1}{a^2}I_n; \ (n = 1; 2; 3; \cdots).$$

Let's find the integrals I_2 and I_3 for $n=1$ and $n=2$ in the formula (12.5):

$$I_2 = \frac{1}{2a^2}\frac{t}{t^2+a^2} + \frac{1}{2a^3}\operatorname{arctg}\frac{t}{a} + C,$$

$$I_3 = \frac{1}{4a^2}\frac{t}{\left(t^2+a^2\right)^2} + \frac{1}{8a^4}\frac{3t}{t^2+a^2} + \frac{3}{8a^5}\operatorname{arctg}\frac{t}{a} + C.$$

Returning to the old variable in the formula $t = x + \frac{p}{2}$, one can find the final results for all integrals I_n.

12.3. The technique of integrating a proper rational fraction

Let it be necessary to find the integral $\int \frac{P_n(x)}{Q_m(x)}dx$. If the fraction $\frac{P_n(x)}{Q_m(x)}$ is improper, then this integral can be presented as the sum of the integral of the polynomial and the proper rational fraction:

$$\int \frac{P_n(x)}{Q_m(x)} = \int W_{n-m}(x)dx + \int \frac{R_p(x)}{Q_m(x)}dx.$$

The integral of a polynomial $W_k(x)$ is found directly, and the integral of a proper rational fraction is reduced to integrals of elementary fractions using formula (12.2).

Conclusion. So, we established that the integration of an arbitrary rational function (fraction) is reduced to the integration of a polynomial and a ***finite***

number of simple fractions, the integrals of which are expressed through ***rational***, ***logarithmic***, and ***inverse trigonometric functions***. Thus, ***any rational function is integrated in elementary functions***.

Let's consider the technique of integrating a proper rational fraction (*examples 12.1 – 12.3*) and an improper rational fraction (*example 12.4*).

Examples 12.6. Find the following indefinite integrals:

a) $\int \frac{2x-3}{x(x-1)(x-2)}dx$; **b)** $\int \frac{x\,dx}{(x-1)(x+1)^2}$; **c)** $\int \frac{x\,dx}{(x-1)(x^2+1)}$;

d) $\int \frac{x^5+2}{x^3-1}dx$; **e)** $\int \frac{2x^3-x^2+8x-1}{x^4+5x^2+4}dx$.

The solution.

a) $\int \frac{2x-3}{x(x-1)(x-2)}dx$. In accordance with formula (12.1), the breakdown of the sum of elementary fractions has the form:

$$\frac{2x-3}{x(x-1)(x-2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-2}.$$

Reducing the fractions to a common denominator (it coincides with the denominator of the given fraction), and equating the numerators of the obtained and given fractions, we will have the identity:

$$2x-3 \equiv A(x-1)(x-2) + Bx(x-2) + C(x-1)x.$$

Let's find the coefficients by the method of individual values of the argument. Let's substitute successively instead of x its individual values $x_1 = 0,\ x_2 = 1,\ x_3 = 2$ in the above identity. We obtained the following system of linear equations: $-3 = 2A,\quad -1 = -B,\quad 1 = 2C.$ From which it follows that $A = -3/2,\ B = 1,\ C = 1/2$. Therefore, the required integral will have the form:

$$\int \frac{2x-3}{x(x-1)(x-2)}dx = \int\left(\frac{-3/2}{x} + \frac{1}{x-1} + \frac{1/2}{x-2}\right)dx = -\frac{3}{2}\ln|x| + \ln|x-1| + \frac{1}{2}\ln|x-2| + C.$$

b) $\int \frac{x\,dx}{(x-1)(x+1)^2}$. The decomposition of a fraction $\frac{x}{(x-1)(x+1)^2}$ into the sum of elementary fractions looks like this:

$$\frac{x}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{(x+1)^2} + \frac{C}{x+1}.$$

Let's reduce the fractions in both parts of the equation to a common denominator. As a result, we will have:

$$x \equiv A(x+1)^2 + B(x-1) + C(x^2-1).$$

With the values $x=1$ and $x=-1$ we find that $4A=1,\ -1=-2B$, that is $A=1/4$, $B=1/2$.

To calculate the value C, we equate the coefficients in the identity with x^2. We obtain the equation $0 = A + C$, from which we find: $C = -\frac{1}{4}$.

Now it is possible to integrate the given rational function:

$$\int \frac{x\,dx}{(x-1)(x+1)^2} = \int \frac{1/4}{x-1}dx + \int \frac{1/2}{(x+1)^2}dx + \int \frac{-1/4}{x+1}dx =$$

$$= \frac{1}{4}\ln|x-1| - \frac{1}{2}\cdot\frac{1}{x+1} - \frac{1}{4}\ln|x+1| + C = \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\frac{1}{x+1} + C.$$

c) $\int \frac{x\,dx}{(x-1)(x^2+1)}$. The decomposition of the fraction into elementary units is as follows:

$$\frac{x}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Mx+N}{x^2+1}.$$

So, $x \equiv A(x^2+1) + (Mx+N)(x-1)$.

When $x=1$ we get $1=2A$, that is, $A = \frac{1}{2}$. Further: $\left.\begin{array}{l|l} x^2 & A+M = 0, \\ x^0 & A-N = 0 \end{array}\right\}.$

From here we get: $M = -\frac{1}{2}$, $N = \frac{1}{2}$. As a result, we have:

$$\int \frac{x\,dx}{(x-1)(x^2+1)} = \int\left(\frac{1/2}{x-1} + \frac{-1/2\,x + 1/2}{x^2+1}\right)dx =$$

$$= \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\operatorname{arctg} x + C.$$

d) $I = \int \frac{x^5+2}{x^3-1}dx$. An ***improper fraction*** is given under the sign of the integral, so first we will allocate its whole part by performing division $(x^5+2):(x^3-1)$.

As a result, we get a polynomial of the second degree and a ***proper fraction***:

$$\frac{x^5+2}{x^3-1} = x^2 + \frac{x^2+2}{x^3-1}.$$

The decomposition of a proper fraction into elementary units looks like this:

$$\frac{x^2+2}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1};$$

$$x^2+2 \equiv A\left(x^2+x+1\right) + \left(Bx+C\right)\cdot\left(x-1\right) \equiv \left(A+B\right)\cdot x^2 + \left(A-B+C\right)\cdot x + \left(A-C\right).$$

We will use the combined method of finding coefficients A, B, C. If $x=1$, then $A=1$. Equating the coefficients at x^2, we get the equation: $A+B=1 \Rightarrow$ where $B=0$. Equating the coefficients at x^0, as a result we will have the following equation: $A-C=2 \Rightarrow C=-1$.

Now you can integrate:

$$I=\int\left(x^2+\frac{1}{x-1}-\frac{1}{x^2+x+1}\right)dx=\frac{x^3}{3}+\ln|x-1|-\int\frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}}=$$

$$=\frac{x^3}{3}+\ln|x-1|-\int\frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{x^3}{3}+\ln|x-1|-\frac{2}{\sqrt{3}}\cdot arctg\,\frac{2x+1}{\sqrt{3}}+C.$$

c) $\int\frac{2x^3-x^2+8x-1}{x^4+5x^2+4}dx$. The integral function is a proper fraction whose denominator has only imaginary roots. Let's decompose this fraction into elementary fractions (intermediate transformations are not shown):

$$\frac{2x^3-x^2+8x-1}{x^4+5x^2+4}=\frac{2x}{x^2+1}-\frac{1}{x^2+4}.$$

Now you can integrate the given function:

$$\int\frac{2x^3-x^2+8x-1}{x^4+5x^2+4}dx=\int\left(\frac{2x}{x^2+1}-\frac{1}{x^2+4}\right)dx=$$
$$=\int\frac{2x}{x^2+1}dx-\int\frac{dx}{x^2+4}=\ln(x^2+1)-\frac{1}{2}\text{arctg}\left(\frac{x}{2}\right)+C.$$

Tasks for classroom and independent work

Find the following indefinite integrals:

12.1. $\int\frac{x^2-x+4}{(x+1)(x-2)(x-3)}dx$.

12.2. $\int\frac{2x^2-1}{x^3-5x^2+6x}dx$.

12.3. $\int\frac{x^2dx}{(x^2-3x+2)^2}$.

12.4. $\int\frac{3x^2+x+3}{(x^2+1)(x-1)^3}dx$.

12.5. $\int\frac{x^4+1}{x^3-x^2+x-1}dx$.

12.6. $\int\frac{x^4+1}{x^6+1}dx$.

12.7. $\int \frac{x^2+5x+4}{x^4+5x^2+4}dx$.

12.8. $\int \frac{x^5+1}{x^3+x^2+x}dx$.

12.9. $\int \frac{5x^2+6x+9}{(x+1)^2(x-3)^2}dx$.

12.10. $\int \frac{x^4+1}{x^4-1}dx$.

12.11. $\int \frac{x^2+1}{x^4+1}dx$.

12.12. $\int \frac{2x+3}{(x+1)^2(x-2)^2}dx$.

12.13. $\int \frac{x^4+2x^3+5x^2+1}{x(x^2+1)^2}dx$.

12.14. $\int \frac{x^3-2x^2+10x+1}{(x^2+4)(x-1)^2}dx$.

12.15. $\int \frac{dx}{(x^2+1)^4}$.

12.16. $\int \frac{x^9dx}{(x^4-1)^2}$.

Answers and instructions

12.1. $\ln \frac{\sqrt{(x+1)(x-3)^5}}{(x-2)^2}+C$.

12.2. $-\frac{1}{6}\ln|x|-\frac{7}{2}\ln|x-2|+\frac{17}{3}\ln|x-3|+C$.

12.3. $-\frac{5x-6}{x^2-3x+2}+4\ln\left|\frac{x-1}{x-2}\right|+C$.

12.4. $\frac{1}{4}\left(\ln\frac{\sqrt{x^2+1}}{|x-1|}+arctgx-\frac{7}{(x-1)^2}\right)+C$.

12.5. $\frac{(x+1)^2}{2}+\ln\frac{|x-1|}{\sqrt{x^2+1}}-arctgx+C$.

12.6. $arctgx+\frac{1}{3}arctgx^3+C$.

12.7. $arctgx+\frac{5}{6}\ln\frac{x^2+1}{x^2+4}+C$.

12.8. $\frac{x^3}{3}-\frac{x^2}{2}+\ln|x|-\frac{2}{\sqrt{3}}arctg\,\frac{2x+1}{\sqrt{3}}+C$.

12.9. $-\frac{9}{2(x-3)}-\frac{1}{2(x+1)}+C$.

12.10. $x+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|-arctgx+C$.

12.11. $\dfrac{\operatorname{arctg}\left(\dfrac{x}{\sqrt{2}}\right)+\operatorname{arctg}\left(\dfrac{x(x^2+1)}{\sqrt{2}}\right)}{\sqrt{2}}+C$.

12.12. $\dfrac{8}{27}\ln\left|\dfrac{x+1}{x-2}\right|-\dfrac{8x+5}{9\cdot(x^2-x-2)}+C$.

12.13. $\dfrac{x+\dfrac{3}{2}}{x^2+1}-\operatorname{arctg}x-\ln x+C$.

12.14. $\ln|x-1|-\dfrac{3}{2}\cdot\operatorname{arctg}\left(\dfrac{x}{2}\right)-\dfrac{2}{x-1}+C$.

12.15. $\dfrac{15x^5+40x^3+33x}{48(x^2+1)^3}+\dfrac{5}{16}\operatorname{arctg}x+C$.

12.16. $\dfrac{1}{4}\left[\dfrac{2x^6-3x^2}{x^4-1}+\dfrac{3}{2}\ln\left|\dfrac{x^2-1}{x^2+1}\right|\right]+C$.

PRACTICAL LESSON 13

Integration of some irrational and trigonometric functions. Trigonometric substitutions. Euler substitutions

13.1. Integration of some irrational functions

We will consider the main types of integrals from irrational functions and show that in some cases they can be reduced to integrals from rational functions (that is, ***they are rationalized***) by appropriate substitutions of variables.

A. ***The integral of the form*** $\int R\left(x;\left(\frac{ax+b}{cx+d}\right)^{r_1/s_1};\cdots;\left(\frac{ax+b}{cx+d}\right)^{r_\alpha/s_\alpha}\right)dx$,

where R – is a known rational function; a, b, c, d – are constants; r_i, s_i – are positive integers, $i=\overline{1,\alpha}$, is reduced to the integral of the rational function of the new variable t using the substitution $\frac{ax+b}{cx+d}=t^m$, where the number m – is the Least Common Multiple (LCM) of the denominators of the fractions $\frac{r_1}{s_1},\cdots,\frac{r_\alpha}{s_\alpha}$, i.e., $m=\mathrm{LCM}(s_1,\cdots,s_\alpha)$. Really:

$$x=\frac{b-d\cdot t^m}{c\cdot t^m-a},\ dx=\frac{mt^{m-1}(ad-bc)}{(ct^m-a)^2}dt\ .$$

Therefore, x and dx are expressed in terms of rational functions from t, and each power of the fraction $\frac{ax+b}{cx+d}$ will be expressed in terms of a whole power of t, then the integrand function will turn into a rational function from t.

In particular, the integral of the form $\int R\left(x;x^{r_1/s_1};\cdots;x^{r_\alpha/s_\alpha}\right)dx$ is reduced to the integral of the rational function of the new variable t by means of substitution $x=t^m$, where

$$m=\mathrm{LCM}(s_1,\cdots,s_\alpha)\ .$$

Examples 13.1. Find integrals:

a) $\int\frac{\sqrt{x}}{\sqrt[4]{x^3}+4}dx$; **b)** $\int\frac{\sqrt[6]{x+1}\,dx}{\sqrt{x+1}+\sqrt[3]{x+1}}$.

The solution.

a) Since $\mathrm{LCM}(2,4)=4$, then

$$\int \frac{\sqrt{x}\,dx}{\sqrt[4]{x^3}+4}=\int \frac{x^{1/2}}{x^{3/4}+4}\,dx=\left|\begin{array}{l} x=t^4 \\ dx=4t^3\,dt \end{array}\right|=4\int \frac{t^2}{t^3+4}t^3\,dt=4\int \frac{t^5}{t^3+4}\,dt=\left|\begin{array}{l|l} t^5 & t^3+4 \\ \cline{2-2} \underline{t^5+4t^2} & t^2 \\ \quad -4t^2 & \end{array}\right|=$$

$$=4\cdot\frac{t^3}{3}-\frac{16}{3}\int\frac{d\left(t^3+4\right)}{t^3+4}=\frac{4}{3}t^3-\frac{16}{3}\ln\left|t^3+4\right|+C=\frac{4}{3}\sqrt[4]{x^3}-\frac{16}{3}\ln\left|\sqrt[4]{x^3}+4\right|+C.$$

b) Since $\mathrm{LCM}(2,3,6)=6$, then

$$\int\frac{\sqrt[6]{x+1}\,dx}{\sqrt{x+1}+\sqrt[3]{x+1}}=\left|\begin{array}{l} x+1=t^6 \\ dx=6t^5\,dt \end{array}\right|=\int\frac{t}{t^3+t^2}6t^5\,dt=6\int\frac{t^4}{t+1}\,dt=$$

$$=6\int\frac{t^4-1+1}{t+1}\,dt=6\int\left(t^3-t^2+t-1+\frac{1}{t+1}\right)dt$$

$$=\frac{3}{2}t^4-2t^3+3t^2-6t+6\ln\left|t+1\right|+C=$$

$$=\frac{3}{2}\sqrt[3]{(x+1)^2}-2\sqrt{x+1}+3\sqrt[3]{x+1}-6\sqrt[6]{x+1}+6\ln\left|\sqrt[6]{x+1}+1\right|+C.$$

B. *Integration of differential binomials.*

Definition. *An expression of the form* $x^r(a+bx^q)^p$, *where* r,q,p – *are constant rational numbers and* a *and* b – *are arbitrary constant numbers, is called a* ***differential binomial****.*

Chebyshev's theorem. *The integral of the differential binomial*

$$\int x^r(a+bx^q)^p\,dx \qquad (13.1)$$

is expressed through the integral of rational functions with respect to the new variable, if:

1) p *is an integer (* $p>0$, *or* $p<0$, *or* $p=0$*) and the substitution* $x=t^s$ *is performed, where* s *is the lowest common denominator of the fractions* r *and* q*;*

2) $\dfrac{r+1}{q}$ *is an integer* $\left(\dfrac{r+1}{q}=0,\dfrac{r+1}{q}>0,\dfrac{r+1}{q}<0\right)$, *and the substitution* $a+bx^q=t^m$ *is made, where* m *is the denominator of the fraction* p*;*

3) $\dfrac{r+1}{q}+p$ *is an integer* $\left(\dfrac{r+1}{q}+p=0,\ \dfrac{r+1}{q}+p<0,\ \dfrac{r+1}{q}+p>0\right)$ *and the substitution* $ax^{-q}+b=t^m$ *is made, where* m *is the denominator of the fraction* p *.*

In addition, Chebyshev's Theorem states that in other cases the integral of the ***differential binomial*** (13.1) ***cannot*** be ***expressed through elementary functions***.

Examples 13.2. Find integrals:

a) $\int x^5 \sqrt[3]{\left(1+x^3\right)^2}\,dx$.

b) $\int x^3(1+2x^2)^{-\frac{3}{2}}dx$.

The solution.

a) $$\int x^5 \sqrt[3]{\left(1+x^3\right)^2}\,dx=\int x^5\left(1+x^3\right)^{\frac{2}{3}}dx=\left|\begin{array}{l} r=5,\ q=3,\ p=\dfrac{2}{3}, \\ \dfrac{r+1}{q}=2\in Z \end{array}\right|=$$

$$=\left|\begin{array}{l} \text{substitution of the 2nd type}: t^3=1+x^3\Rightarrow \\ \Rightarrow x=\sqrt[3]{t^3-1}\Rightarrow dx=t^2\left(t^3-1\right)^{-\frac{2}{3}}dt \end{array}\right|=$$

$$=\int\left(t^3-1\right)^{\frac{5}{3}}\cdot t^2\cdot t^2\cdot\left(t^3-1\right)^{-\frac{2}{3}}dt=\int\left(t^3-1\right)t^4dt=\frac{t^8}{8}-\frac{t^5}{5}+C=$$

$$=\frac{1}{8}\left(1+x^3\right)^{\frac{8}{3}}-\frac{1}{5}\left(1+x^3\right)^{\frac{5}{3}}+C.$$

b) $$\int x^3(1+2x^2)^{-\frac{3}{2}}dx=\left|\begin{array}{l} \text{substitution of the 2nd type};\ r=3;q=2;\ p=-\dfrac{3}{2};\dfrac{r+1}{q}=2; \\ \Rightarrow 1+2x^2=t^m\Rightarrow m=2\Rightarrow x=\sqrt{\dfrac{t^2-1}{2}};dx=\dfrac{tdt}{\sqrt{2(t^2-1)}} \end{array}\right|=$$

$$=\int\left(\frac{t^2-1}{2}\right)^{\frac{3}{2}}t^{-3}\cdot\frac{tdt}{\sqrt{2(t^2-1)}}=\frac{1}{4}\int\frac{t^2-1}{t^2}dt=\frac{1}{4}\left(t+\frac{1}{t}\right)+C=\frac{1+x^2}{2\sqrt{1+2x^2}}+C.$$

13.2. Integration of trigonometric functions

Let's establish some types of integrals from trigonometric functions that are integrated in a ***closed form***. They include integrals of rational functions with respect to functions $\sin x, \cos x, \operatorname{tg} x, \operatorname{ctg} x, \sec x, \operatorname{cosec} x$.

I. First, consider the ***general case*** in which the ***universal trigonometric substitution*** is used.

Consider integrals of the form $\int R(\sin x, \cos x)dx$, where $R -$ is a rational function with respect to $\sin x$ and $\cos x$. This function (and the integral with it) is ***always rationalized*** by the universal substitution $\operatorname{tg}\frac{x}{2} = t$. Really,

$$\sin x = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-t^2}{1+t^2};$$

$$x = 2\operatorname{arctg} t; \qquad dx = \frac{2dt}{1+t^2}.$$

Thus, the universal trigonometric substitution has the form:

$$\operatorname{tg}\frac{x}{2} = t; \quad x = 2\operatorname{arctg} t; \quad \sin x = \frac{2t}{1+t^2}; \quad \cos x = \frac{1-t^2}{1+t^2}; \quad dx = \frac{2dt}{1+t^2}.$$

Therefore,

$$\int R(\sin x; \cos x)dx = \int R\left(\frac{2t}{1+t^2}; \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2} = \int R^*(t)dt,$$

where $R^*(t)$ is a known rational function from t.

Remark. This substitution often leads to rational fractions with large powers, so in many cases other substitutions are used. We will list some of them.

II. Consider partial cases of integrals of trigonometric functions, which are rationalized ***by other substitutions***.

1. Integrals of the form $I_1 = \int R(\sin x)\cos x dx$ are rationalized by replacing the variable $t = \sin x$; $dt = \cos x dx$; $x = \arcsin t$; $dx = \frac{dt}{\sqrt{1-t^2}}$. Then $I_1 = \int R_1(t)dt$.

2. Integrals of the form $I_2 = \int R(\cos x)\sin x dx$ are rationalized by replacing the variable $t = \cos x$; $dt = -\sin x dx$; $x = \arccos t$; $dx = -\frac{dt}{\sqrt{1-t^2}}$. Then $I_2 = \int R_2(t)dt$.

3. Integrals of the form $I_3 = \int R(\operatorname{tg} x)\,dx$ are rationalized by replacing the variable $t = \operatorname{tg} x;\ x = \operatorname{arctg} t;\ dx = \frac{dt}{1+t^2}$. Then $I_3 = \int R(t)\cdot\frac{dt}{1+t^2} = \int R_3(t)\,dt$. For example, the integral $\int \operatorname{tg}^7 x dx$ is reduced to the integration of the following improper rational fraction: $I_3 = \int \operatorname{tg}^7 x dx = \left|t = \operatorname{tg} x\right| = \int \frac{t^7 dt}{1+t^2}$.

In these cases, $R_1(t), R_2(t), R_3(t)$ – the corresponding ***known rational functions*** from t.

4. For integrals of the form $I_4 = \int R(\sin x, \cos x)\,dx$, the following substitutions are used:

a) if $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, that is, the function $R(\sin x, \cos x)$ is ***odd*** relative to $\sin x$, then this integral is rationalized by substitution $\cos x = t$.

b) if $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, that is, the function $R(\sin x, \cos x)$ is ***odd*** relative to $\cos x$, then this integral is rationalized by substitution $\sin x = t$.

c) if $R(-\sin x, -\cos x) = R(\sin x, \cos x)$, that is, the function $R(\sin x, \cos x)$ ***is even relative to*** $\sin x$ ***and*** $\cos x$ ***at the same time***, then this integral is rationalized by substitution $\operatorname{tg} x = t$. For example, the integral $\int \sin^{-\frac{2}{3}} x \cdot \cos^{-\frac{4}{3}} x dx$.

5. Depending on the numbers m and n, integrals of the form $I_5 = \int \sin^m x \cos^n dx$ are found using the following substitutions:

a) if n is a ***positive odd integer***,then this integral is rationalized by substitution $\sin x = t$;

b) if m is a ***positive odd integer***, then this integral is rationalized by the substitution $\cos x = t$;

c) if m and n are ***even positive integers***, then this integral is found using the formulas for ***decreasing the power***:

$$\cos^2 x = \frac{1+\cos 2x}{2}; \quad \sin^2 x = \frac{1-\cos 2x}{2};$$

d) if m and n are ***even integers***, but at least one of them is negative, then this integral is rationalized by substitution $\operatorname{tg} x = t$. The same substitution is used in the case when m and n are ***odd and negative integers***. For example, such integrals are found using substitution $\operatorname{tg} x = t$:

$$I_1 = \int \frac{\sin^4 x}{\cos^8 x} dx = \left|t = \operatorname{tg} x\right| = \int t^4(1+t^2)dt = \frac{t^5}{5} + \frac{t^7}{7} + C = \frac{\operatorname{tg}^5 x}{5} + \frac{\operatorname{tg}^7 x}{7} + C;$$

$$I_2 = \int \frac{dx}{\sin x \cos^3 x} = \left|t = \operatorname{tg} x\right| = \int \frac{1+t^2}{t} dt = \ln|t| + \frac{t^2}{2} + C = \ln\left|\operatorname{tg} x\right| + \frac{\operatorname{tg}^2 x}{2} + C.$$

6. Integrals of the form $I_7 = \int \sin ax \sin bx dx$, $I_8 = \int \cos ax \cos bx dx$ are calculated using the following "school" formulas:

$$\sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right];$$

$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right];$$

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right].$$

Examples 13.3. Find Integrals:

a) $\int \frac{dx}{1+\sin x+\cos x}$;

b) $\int \frac{dx}{3+\sin^2 x}$;

c) $\int \mathrm{tg}^5\, 2x\, dx$;

d) $\int \frac{\sin^3 x}{\cos^4 x} dx$;

e) $\int \sin^4 x \cdot \cos^5 x dx$;

f) $\int \sin^2 x \cdot \cos^4 x dx$;

g) $\int \sin 5x \cdot \cos 3x dx$.

The solution.

a) $\int \frac{dx}{1+\sin x+\cos x} = \left| \begin{matrix} \mathrm{tg}\frac{x}{2} = t;\ x = 2arctgt; \\ dx = \frac{2}{1+t^2}dt \end{matrix} \right| =$

$$= \int \frac{2\,dt/(1+t^2)}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{dt}{1+t} = \ln|1+t| + C = \ln\left|1+\mathrm{tg}\frac{x}{2}\right| + C.$$

b) $\int \frac{dx}{3+\sin^2 x} = \left| \begin{matrix} \mathrm{tg}\, x = t;\ \sin^2 x = \frac{t^2}{1+t^2}; \\ x = \mathrm{arctg}\, t;\ dx = \frac{dt}{1+t^2} \end{matrix} \right| = \int \frac{dt/(1+t^2)}{3+t^2/(1+t^2)} = \int \frac{dt}{3+4t^2} =$

$$= \frac{1}{2\sqrt{3}}\mathrm{arctg}\frac{2t}{\sqrt{3}} + C = \frac{1}{2\sqrt{3}}\mathrm{arctg}\frac{2tg\ x}{\sqrt{3}} + C.$$

c) $\int \mathrm{tg}^5 2x dx = \left| t = \mathrm{tg}\, 2x;\ x = \frac{1}{2}\mathrm{arctg}\, t;\ dx = \frac{1}{2}\frac{1}{t^2+1}dt \right| = \frac{1}{2}\int t^5 \frac{dt}{t^2+1} =$

$= \frac{1}{2}\int \left(t^3 - t + \frac{t}{1+t^2} \right) dt = \frac{1}{8}t^4 - \frac{1}{2}t^2 + \frac{1}{4}\ln(1+t^2) + C =$

$= \frac{1}{8}\mathrm{tg}^4 2x - \frac{1}{4}\mathrm{tg}^2 2x + \frac{1}{4}\ln(1+\mathrm{tg}^2 2x) + C.$

d) $\int \frac{\sin^3 x}{\cos^4 x} dx = \int \frac{1-\cos^2 x}{\cos^4 x}\sin x\, dx = \begin{vmatrix} \cos x = t \\ -\sin x\, dx = dt \end{vmatrix} = \int \frac{1-t^2}{t^4}(-dt) =$

$= -\int \frac{1}{t^4}dt + \int \frac{1}{t^2}dt = \frac{1}{3}t^{-3} - \frac{1}{t} + C = \frac{1}{3}\frac{1}{\cos^3 x} - \frac{1}{\cos x} + C.$

e) $\int \sin^4 x \cdot \cos^5 x dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x dx =$

$= \int \sin^4 x \cdot (1-\sin^2 x)^2 \cdot \cos x dx = \begin{vmatrix} \sin x = t \\ \cos x dx = dt \end{vmatrix} = \int t^4 (1-t^2)^2 dt =$

$= \int (t^4 - 2t^6 + t^8)\, dt = \frac{1}{5}t^5 - \frac{2}{7}t^7 + \frac{t^9}{9} + C = \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$.

f) $\int \sin^2 x \cdot \cos^4 x dx = \frac{1}{2}\int (\sin x \cdot \cos x)^2 \cdot \cos^2 x dx =$

$= \int \left(\frac{\sin 2x}{2} \right)^2 \cdot \frac{1+\cos 2x}{2}\, dx = \frac{1}{8}\int \sin^2 2x dx + \frac{1}{8}\int \sin^2 2x \cdot \cos 2x dx =$

$= \frac{1}{16}\int (1-\cos 4x)\, dx + \frac{1}{2\cdot 8}\int \sin^2 2x \cdot d(\sin 2x) =$

$= \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C$.

g) $\int \sin 5x \cdot \cos 3x dx = \frac{1}{2}\int [\sin(5-3)x + \sin(5+3)x]\, dx =$

$= \frac{1}{2}\int \sin 2x dx + \frac{1}{2}\int \sin 8x dx = -\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C$.

13.3. Trigonometric substitutions

Consider integrals of the following form:

$$\int R\left(x; \sqrt{ax^2 + bx + c}\right) dx,$$

where $R\left(x; \sqrt{ax^2 + bx + c}\right)$ is some rational function.

We will show that the specified integrals can be reduced to one of three integrals using substitution $\left| x = t - \frac{b}{2a};\ x + \frac{b}{2a} = t;\ dx = dt \right|$:

a) $\int R\left(t;\sqrt{m^2-t^2}\right)dt$; b) $\int R\left(t;\sqrt{t^2+m^2}\right)dt$; c) $\int R\left(t;\sqrt{t^2-m^2}\right)dt$.

Let's isolate the complete square in the quadratic trinomial under the root

$$ax^2+bx+c=a\left(x^2+\frac{b}{a}x\right)+c=a\left[x^2+2x\cdot\frac{b}{2a}+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right]+c=$$

$$=a\left[\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right]=a\left[\left(x+\frac{b}{2a}\right)^2\pm m^2\right].$$

If $a>0$, we have: $\int R\left(x;\sqrt{ax^2+bx+c}\right)dx=\int R\left(t;\sqrt{t^2\pm m^2}\right)dt$.

If $a<0$, we get: $\int R\left(x;\sqrt{ax^2+bx+c}\right)dx=\int R\left(t;\sqrt{m^2-t^2}\right)dt,\ m\in R_1$.

To find integrals of the form a), b), c) use trigonometric substitutions that reduce the given integrals to integrals of the form $R(\sin z,\cos z)$.

For the integral $\int R\left(t;\sqrt{m^2-t^2}\right)dt$ use the substitution $t=m\sin z$, or $t=m\cos z$.

For the integral $\int R\left(t;\sqrt{t^2-m^2}\right)dt$ use the substitution $t=\dfrac{m}{\sin z}$, or $t=\dfrac{m}{\cos z}$.

For the integral $\int R\left(t;\sqrt{t^2+m^2}\right)dt$ use the substitution $t=m\cdot\operatorname{tg} z$, or $t=m\cdot\operatorname{ctg} z$.

Examples 13.4. Find the integrals:

a) $I=\int\sqrt{a^2-x^2}\,dx,\ a>0$;

b) $I=\int\dfrac{dx}{x^2\sqrt{x^2-9}}$;

c) $I=\int\dfrac{dx}{\sqrt{(x^2+9)^3}}$.

The solution.

a) $I=\int\sqrt{a^2-x^2}\,dx,\ a>0$. If you put $x=a\sin t$, then $\sqrt{a^2-x^2}=a\sqrt{1-\sin^2 t}=a\cos t$ and $dx=a\cos t\,dt$. So, due to the new variable, the integral looks like this:

$$I=\int a\cos t\cdot a\cos t\,dt=a^2\int\cos^2 t\,dt=a^2\int\frac{1+\cos 2t}{2}dt=\frac{a^2t}{2}+\frac{a^2}{4}\sin 2t+C.$$

Returning to the old variable using the formula $t = \arcsin\frac{x}{a}$, we obtain:

$$I = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2} + C = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C.$$

Therefore, the integral has the final form:

$$I = \int\sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C .$$

b) $I = \int\frac{dx}{x^2\sqrt{x^2-9}}$. Here we will apply the replacement: $x = \frac{3}{\sin t}$. So, as a result of this substitution, we get:

$$I = \int\frac{dx}{x^2\sqrt{x^2-9}} = \left| \begin{array}{l} x = \frac{3}{\sin t};\; dx = -\frac{3dt\cos t}{\sin^2 t}; \\ \sqrt{x^2-9} = \frac{3\cos t}{\sin t};\; \cos t = \frac{1}{x}\sqrt{x^2-9} \end{array} \right| = \int\frac{-\frac{3dt\cos t}{\sin^2 t}}{\frac{9}{\sin^2 t}\cdot\frac{3\cos t}{\sin t}} =$$

$$= -\frac{1}{9}\int\sin t dt = \frac{1}{9}\cos t + C = \frac{1}{9x}\sqrt{x^2-9} + C .$$

c) $I = \int\frac{dx}{\sqrt{(x^2+9)^3}}$. Here we will apply the replacement: $x = 3\,\mathrm{tg}\,t$. As a result of this substitution, we get:

$$I = \int\frac{dx}{\sqrt{(x^2+9)^3}} = \left| \begin{array}{l} x = 3\,\mathrm{tg}\,t;\; dx = \frac{3dt}{\cos^2 t} \\ \sqrt{(x^2+9)^3} = \frac{27}{\cos^3 t}. \end{array} \right| = \int\frac{\frac{3dt}{\cos^2 t}}{\frac{27}{\cos^3 t}} = \frac{1}{9}\int\cos t dt = \frac{1}{9}\sin t + C =$$

$$= \left| \begin{array}{l} \sin t = \frac{\mathrm{tg}\,t}{\sqrt{1+\mathrm{tg}^2 t}}; \\ \mathrm{tg}\,t = \frac{x}{3} \end{array} \right| = \frac{1}{9}\frac{\frac{x}{3}}{\sqrt{1+\left(\frac{x}{3}\right)^2}} + C = \frac{1}{9}\frac{x}{\sqrt{(x^2+9)}} + C .$$

13.4. Euler substitutions

In some cases, it is more convenient to use Euler substitutions for integrals of the form $\int R\left(x;\sqrt{ax^2+bx+c}\right)dx$. They also ***rationalize*** the integrand, but without using trigonometric functions. So, Euler substitutions have the following form:

The first substitution: if $a > 0$, then $\sqrt{ax^2+bx+c} = \pm\sqrt{a}\,x + t$.

The second substitution: if $c > 0$, then $\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}$.

The third substitution: if $b^2 - 4ac > 0$, then $\sqrt{ax^2 + bx + c} = (x - k)t$, where $k -$ the root of the trinomial $ax^2 + bx + c$.

We will give three examples of the use of Euler substitutions.

Example 13.5. ***The first substitution*** ($a = 1, a > 0$).

The solution.

$$J = \int \frac{dx}{x\sqrt{x^2 + 2x + 9}} = \left| \begin{array}{l} t = \sqrt{x^2 + 2x + 9} + x;\ x = \dfrac{t^2 - 9}{2(1 + t)}; \\ dx = \dfrac{t^2 + 2t + 9}{2(t + 1)} dt; \\ \sqrt{x^2 + 2x + 9} = t - x = \dfrac{t^2 + 2t + 9}{2(t + 1)} \end{array} \right| == \int \frac{\dfrac{t^2 + 2t + 9}{2(t + 1)} dt}{\dfrac{t^2 - 9}{2(1 + t)} \cdot \dfrac{t^2 + 2t + 9}{2(t + 1)}} =$$

$$= \int \frac{2dt}{t^2 - 9} = \frac{1}{3} \ln \left| \frac{t - 3}{t + 3} \right| + C = \frac{1}{3} \ln \left| \frac{\sqrt{x^2 + 2x + 9} + x - 3}{\sqrt{x^2 + 2x + 9} + x + 3} \right| + C .$$

Example 13.6. ***The second substitution*** ($c = 1, c > 0$).

The solution.

$$J = \int \frac{dx}{(1 + x)\sqrt{1 - x - x^2}} = \left| \begin{array}{l} \sqrt{1 - x - x^2} = xt - 1; \\ 1 - x - x^2 = x^2 t^2 - 2xt + 1; \\ x = \dfrac{2t - 1}{1 + t^2};\ dx = \dfrac{2(1 + t - t^2)}{(1 + t^2)^2} dt \end{array} \right| =$$

$$= -\int \frac{2dt}{t^2 + 2t} = -2\int \left(\frac{1}{t + 2} - \frac{1}{t} \right) + C = -2 \ln \left| 1 + \frac{2}{t} \right| + C =$$

$$= \left| t = \frac{\sqrt{1 - x - x^2} + 1}{x} \right| = -2 \ln \left| 1 + \frac{2x}{\sqrt{1 - x - x^2} + 1} \right| + C .$$

Example 13.7. ***The third substitution*** ($b^2 - 4ac > 0$).

The solution.

$$J = \int \frac{dx}{(1 + x)\sqrt{x^2 - 5x + 4}} = \left| \begin{array}{l} \sqrt{x^2 - 5x + 4} = (x - 1)t; \\ \dfrac{x - 4}{x - 1} = t^2;\ x = \dfrac{t^2 - 4}{t^2 - 1}; \\ dx = \dfrac{6tdt}{(t^2 - 1)^2}; \end{array} \right| = \int \frac{\dfrac{6tdt}{(t^2 - 1)^2}}{\left(1 + \dfrac{t^2 - 4}{t^2 - 1} \right)\left(\dfrac{t^2 - 4}{t^2 - 1} - 1 \right) t} =$$

$$= -\int \frac{dt}{\left(t^2 - \frac{5}{2}\right)} = -\frac{1}{\sqrt{10}} \ln\left|\frac{\sqrt{2}t - \sqrt{5}}{\sqrt{2}t + \sqrt{5}}\right| + C =$$

$$= \left| t = \sqrt{\frac{x-4}{x-1}} \right| = -\frac{1}{\sqrt{10}} \ln\left|\frac{\sqrt{2(x-4)} - \sqrt{5(x-1)}}{\sqrt{2(x-4)} + \sqrt{5(x-1)}}\right| + C .$$

13.5. Integration of functions of the form $R(e^x)$

Integration of functions of the form $R(e^x)$ are rationalized by the ***substitution*** $t = e^x$. After replacing the variable, the process of integrating such functions is not difficult. Let's show it. Since $x = \ln t$, then $dx = \frac{dt}{t}$. As a result of the substitution, we have:

$$\int R(e^x)dx = \int R(t)\frac{dt}{t} .$$

Examples 13.8. Find the integral: $J = \int \frac{3e^{2x} + 4e^x - 3}{e^{2x} + 2e^x - 3}dx$.

The solution.

$$J = \int \frac{3e^{2x} + 4e^x - 3}{e^{2x} + 2e^x - 3}dx = \left| \begin{array}{l} t = e^x;\ x = \ln t; \\ dx = \frac{dt}{t}; \end{array} \right| = \int \frac{3t^2 + 4t - 3}{t(t-1)(t+3)}dt = \left| \begin{array}{l} \text{the method of undetermined} \\ \text{coefficients gives the schedule:} \\ \frac{3t^2 + 4t - 3}{t(t-1)(t+3)} = \frac{1}{t} + \frac{1}{t-1} + \frac{1}{t+3} \end{array} \right| =$$

$$= \int \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t+3}\right)dt == \ln|t| + \ln|t-1| + \ln|t+3| + C =$$

$$= x + \ln|e^x - 1| + \ln|e^x + 3| + C = x + \ln|e^{2x} + 2e^x - 3| + C .$$

Tasks for classroom and independent work

Find the given indefinite integrals:

13.1. $\int \frac{1}{(x-1)^3} \cdot \sqrt[3]{\frac{x+1}{x-1}}dx$.

13.2. $\int \frac{1 - \sqrt{x+1}dx}{1 + \sqrt[3]{x+1}}$.

13.3. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$.

13.4. $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx$.

13.5. $\int x^2 \cdot \sqrt{\frac{x}{1-x}}dx$.

13.6. $\int \sqrt{\frac{1-x}{1+x}} \cdot \frac{dx}{x^2}$.

13.7. $\int \frac{xdx}{(x+1)\sqrt{1-x-x^2}}$.

13.8. $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$.

13.9. $\int \mathrm{ctg}^3\, xdx$.

13.10. $\int \sin^6 xdx$.

13.11. $\int \frac{dx}{\cos^4 x}$.

13.12. $\int \frac{\cos^4 x}{\sin^3 x}dx$.

13.13. $\int \cos 5x \cos 4xdx$.

13.14. $\int \sin 3x \cos 5xdx$.

13.15. $\int \frac{dx}{\sin^2 x \cos^4 x}$.

13.16. $\int \frac{\sin^2 x}{1+\sin^2 x}dx$.

13.17. $\int \frac{dx}{8-4\sin x+7\cos x}$.

13.18. $\int \frac{dx}{2\sin x-\cos x+5}$.

13.19. $\int \frac{dx}{4+5\sin^2 x-3\cos^2 x}$.

13.20. $\int \frac{dx}{1+tgx}$.

13.21. $\int \frac{\sqrt{x^2-1}dx}{x}$.

13.22. $\int \frac{x^2dx}{\sqrt{x^2+9}}$.

13.23. $\int \frac{x^2dx}{\sqrt{4-x^2}}$.

13.24. $\int \frac{dx}{1+\sqrt{1-2x-x^2}}$.

13.25. $\int \frac{dx}{x+\sqrt{x^2+x+1}}$.

13.26. $\int \frac{dx}{x\sqrt{2+x-x^2}}$.

13.27. $\int \frac{dx}{e^{2x}+e^x-2}$.

13.28. $\int \frac{1+e^{x/2}}{\left(1+e^{x/4}\right)^2}dx$.

13.29. $\int \frac{e^{6x}dx}{2\,\mathrm{ch}\,x}$.

13.30. $\int \frac{\mathrm{arctg}(e^{2x})dx}{e^{2x}}$.

13.31. $\int \frac{dx}{\cos^4 x+\sin^4 x}$.

13.32. $\int \frac{\mathrm{tg}\, xdx}{1+\mathrm{tg}\, x+\mathrm{tg}^2\, x}$.

Answers and instructions

13.1. $\frac{3}{16}\sqrt[3]{\left(\frac{x+1}{x-1}\right)^4}-\frac{3}{28}\sqrt[3]{\left(\frac{x+1}{x-1}\right)^7}+C$.

13.2. $6t-3t^2-2t^3+\frac{3}{2}t^4+\frac{6}{5}t^5-\frac{6}{7}t^7+3\ln(1+t^2)-6arctgt+C,\ t=\sqrt[6]{x+1}.$

13.3. $(1+\sqrt[4]{x})\sqrt[3]{1+\sqrt[4]{x}}\left(\frac{12}{7}(1+\sqrt[4]{x})-3\right)+C$.

13.4. $\frac{6}{5}x^{5/6}-4\sqrt{x}+18x^{1/6}+\frac{3x^{1/6}}{1+\sqrt[3]{x}}-21arctgx^{1/6}+C$.

13.5. $-\frac{1}{24}(15+10x+8x^2)\sqrt{x(1-x)}+\ +\frac{5}{8}\arcsin\sqrt{x}+C$.

13.6. $\ln\left|\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1-x}-\sqrt{1+x}}\right|-\frac{\sqrt{1-x^2}}{x}+C$.

13.7. $\ln\left|\frac{3+x+2\sqrt{1-x-x^2}}{1+x}\right|+\arcsin\frac{1+2x}{\sqrt{5}}+C.$

13.8. $-\ln\left|\frac{2-x+2\sqrt{x^2+x+1}}{x+1}\right|+C$.

13.9. $-\frac{1}{2}ctg^2x-\ln|\sin x|+C$.

13.10. $\frac{5}{16}x-\frac{1}{4}\sin 2x+\frac{3}{64}\sin 4x+\frac{1}{48}\sin^3 2x+C$.

13.11. $\frac{1}{3}tg^3x+tgx+C$.

13.12. $-\frac{3}{2}\cos x-\frac{\cos^3 x}{2\sin^2 x}-\frac{3}{2}\ln\left|tg\frac{x}{2}\right|+C$.

13.13. $\frac{1}{18}\sin 9x+\frac{1}{2}\sin x+C$.

13.14. $-\frac{1}{16}cos8x+\frac{1}{4}\cos 2x+C$.

13.15. $tgx+\frac{1}{3}tg^3x-2ctgx+x+C$.

13.16. $x-\frac{1}{\sqrt{2}}arctg(\sqrt{2}tgx)+C$. **13.17.** $\ln\left|\frac{tg\frac{x}{2}-5}{tg\frac{x}{2}-3}\right|+C$.

13.18. $\frac{1}{\sqrt{5}}arctg\left(\frac{3tg\frac{x}{2}+1}{\sqrt{5}}\right)+C$.

13.19. $\frac{1}{3}arctg(3tgx)+C$. **13.20.** $\frac{1}{2}(x+\ln|\sin x+\cos x|)+C$.

13.21. $\sqrt{x^2-1}-\arccos\frac{1}{x}+C$.

13.22. $\frac{x}{2}\sqrt{x^2+9}-\frac{9}{2}\ln(x+\sqrt{x^2+9})+C$. **13.23.** $2\arcsin\frac{x}{2}-\frac{x}{2}\sqrt{4-x^2}+C$.

13.24. $\ln\left|\frac{z-1}{z}\right|-2arctgz+C,\ z=\frac{1+\sqrt{1-2x-x^2}}{x}$.

13.25. $\frac{3}{2(2z+1)}+\frac{1}{2}\ln\frac{z^4}{|2z+1|^3}+C,\quad z=x+\sqrt{x^2+x+1}$.

13.26. $-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x}+\frac{\sqrt{2}}{4}\right|+C$.

13.27. $-\frac{x}{2}+\frac{1}{3}\ln|e^x-1|+\frac{1}{6}\ln(e^x+2)+C$.

13.28. $x+\frac{8}{1+e^{x/4}}+C$. **13.29.** $\frac{e^{5x}}{5}-\frac{e^{3x}}{3}-\mathrm{arctg}(e^x)+e^x+C$.

13.30. $x-\frac{\ln(e^{4x}+1)}{4}-\mathrm{arctg}(e^{2x})+C$. **13.31.** $\frac{1}{\sqrt{2}}\mathrm{arctg}\left(\frac{\mathrm{tg}(2x)}{\sqrt{2}}\right)+C$.

13.32. $x-\frac{2}{\sqrt{3}}\mathrm{arctg}\left(\frac{1+2\,\mathrm{tg}\,x}{\sqrt{3}}\right)+C$.

Topic II. The definite integral and its practical application

PRACTICAL LESSON 14

The Newton-Leibniz formula and other methods of integrating the definite integral. The definite integral as a function of the variable upper bound

14.1. The technique of constructing the Riemann integral. Application of the Newton–Leibniz formula

To understand the technique of constructing the Riemann integral, consider the concept of the limit of integral sums using the following example.

Example. Calculate the integral $\int_0^1 x^2 dx$ as a limit of integral sums.

The solution. We have $f(x) = x^2$, $a = 0$, $b = 1$. Let's divide the segment $[0,1]$ of integration into n equal parts, then $\Delta x_k = \frac{b-a}{n} = \frac{1}{n}$, and choose the partition points as follows: $\xi_k = x_k$. We have:

$$x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, \cdots, x_{n-1} = \frac{n-1}{n}, x_n = 1;$$

$$f(\xi_0) = 0,\ f(\xi_1) = \left(\frac{1}{n}\right)^2,\ f(\xi_2) = \left(\frac{2}{n}\right)^2, \cdots,\ f(\xi_{n-1}) = \left(\frac{n-1}{n}\right)^2,\ f(\xi_n) = 1;$$

$$f(\xi_k)\Delta x_k = \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n}.$$

Thus, by the definition of the definite integral in the Riemann sense, we have:

$$\int_0^1 x^2 dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^2 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^3} = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3}.$$

In the last transformations, the formula of the sum of squares of natural numbers was used.

Next, we present the ***key Newton–Leibniz formula*** in integral calculus.

Theorem 14.1 (*Newton–Leibniz formula*). If $F(x)$ – any ***original function*** from a continuous function $f(x)$, then the formula is valid:

$$\int_a^b f(x)dx = F(b) - F(a)\,. \qquad (14.1)$$

Let's give a few examples using the Newton–Leibniz formula.

Example 14.1. Calculate definite integrals:

a) $\int_0^1 \frac{x + \operatorname{arctg} x}{1 + x^2} dx$;

b) $\int_0^{\pi/6} \frac{\cos x}{4\sin^2 x + 1} dx$;

c) $\int_0^{\pi/4} \sin 2x \cdot e^{\sin^2 x} dx$;

d) $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx$.

The solution.

a) $$\int_0^1 \frac{x + arctgx}{1 + x^2} dx = \int_0^1 \frac{x}{1 + x^2} dx + \int_0^1 \frac{arctgx}{1 + x^2} dx = \frac{1}{2}\int_0^1 \frac{d(x^2 + 1)}{x^2 + 1} + \int_0^1 arctgxd\left(arctgx\right) =$$

$$= \frac{1}{2}\ln\left(x^2 + 1\right)\Big|_0^1 + \frac{arctg^2 x}{2}\Bigg|_0^1 = \frac{1}{2}\ln 2 + \frac{\pi^2}{32}.$$

b) $$\int_0^{\pi/6} \frac{\cos x}{4\sin^2 x + 1} dx = \frac{1}{4}\int_0^{\pi/6} \frac{d(\sin x)}{\sin^2 x + (0,5)^2} = \frac{1}{2}\operatorname{arctg}(2\sin x)\Big|_0^{\pi/6} = \frac{\pi}{8}.$$

c) $$\int_0^{\pi/4} \sin 2x \cdot e^{\sin^2 x} dx = \int_0^{\pi/4} e^{\sin^2 x} d(\sin^2 x) = e^{\sin^2 x}\Big|_0^{\pi/4} = \sqrt{e} - 1.$$

d) $$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2} dx = -\int_1^2 e^{\frac{1}{x}} d\left(\frac{1}{x}\right) = -e^{\frac{1}{x}}\Bigg|_1^2 = e - \sqrt{e}.$$

14.2. Replacement of a variable in the definite integral

When calculating definite integrals, the ***variable replacement (substitution) method*** is very often used.

Theorem 14.2 (***On the replacement of a variable in a definite integral*** $x = \varphi(t)$).

Let the following conditions be fulfilled:

1) the function $f(x)$ is ***continuous*** on the segment $[a,b]$;

2) the function $x = \varphi(t)$ and its derivative $x' = \varphi'(t)$ are ***continuous*** on the segment $[\alpha, \beta]$;

3) $\varphi(\alpha) = a$; $\varphi(\beta) = b$ and $\forall t \in (\alpha, \beta)$: $a < \varphi(t) < b$. Then the following

relation holds:

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt. \qquad (14.2)$$

Remark 14.1. If, when finding the ***indefinite integral*** using variable substitution $x = \varphi(t)$ in the original function, it was necessary to return from the new variable t to the old variable x, then when calculating the ***definite integral***, it is necessary to ***change the limits of integration instead, and the obtained result of integration is final***.

Remark 14.2. New limits of integration (lower and upper) are found from the following equations: $a = \varphi(\alpha)$ and $b = \varphi(\beta)$. Hence we have: $\alpha = \varphi^{-1}(a)$ and $\beta = \varphi^{-1}(b)$.

Remark 14.3. If the replacement of the variable in the definite integral is performed by the formula $t = \psi(x)$, then the new limits α and β can be determined directly by the following formulas: $\alpha = \psi(a)$, $\beta = \psi(b)$.

However, it should be borne in mind here that the function $x = \varphi(t)$, which is the inverse of $t = \psi(x)$, must satisfy all the conditions of Theorem 14.2. In particular, the function $x = \varphi(t)$ within the limits of integration $[\alpha, \beta]$ must be a ***continuously differentiable function*** of the argument t, and when it changes from α to β, the function $x(t)$ must change from a to b.

It is most convenient to replace the variable with functions that are ***monotonically differentiable*** on the integration interval. This property guarantees the uniqueness of both direct and inverse functions.

Example 14.2. Calculate definite integral: $\int_0^{0,5} x^2\sqrt{1-x^2}\,dx$.

The solution. Let's replace the variable: $x = \sin t$. The ***new limits*** of ***integration*** can be found from the equations $\sin\alpha = 0$ and $\sin\beta = \frac{1}{2}$. We have $\alpha = 0,\ \beta = \frac{\pi}{6}$. When t changing from 0 to $\frac{\pi}{6}$, the variable $x = \sin t$ will pass the entire integration interval $[0, 1/2]$. As a result, we get:

$$\int_0^{0,5} x^2\sqrt{1-x^2}\,dx = \int_0^{\pi/6} \sin^2 t\cos^2 t\,dt = \frac{1}{4}\int_0^{\pi/6} \sin^2 2t\,dt =$$

$$= \frac{1}{8}\int_0^{\pi/6} (1-\cos 4t)dt = \left(\frac{t}{8} - \frac{\sin 4t}{32}\right)\Bigg|_0^{\pi/6} = \frac{\pi}{48} - \frac{\sqrt{3}}{64}.$$

Example 14.3. Calculate definite integral: $\int_0^3 x\sqrt{1+x}dx$.

The solution. Let's replace the variable: $t=\sqrt{1+x}$, $x=t^2-1$ and $dx=2tdt$. From the formula $t=\sqrt{1+x}$, we find new limits of integration: $x=0\Rightarrow t=1$ and $x=3\Rightarrow t=2$. After integration, we get:

$$\int_0^3 x\cdot\sqrt{1+x}\cdot dx=\int_1^2\left(t^2-1\right)\cdot t\cdot 2t\cdot dt=2\int_1^2\left(t^4-t^2\right)\cdot dt=2\left(\frac{t^5}{5}-\frac{t^3}{3}\right)\Bigg|_1^2=7\frac{11}{15}.$$

Example 14.4. Calculate definite integral: $\int_1^{\sqrt{3}}\frac{dx}{x^4\cdot\sqrt{4-x^2}}$.

The solution. Let's make a variable substitution: $x=2\cos t$; $dx=-2\sin tdt$; $t=\arccos\left(\frac{x}{2}\right)$; $t_1=\frac{\pi}{3};t_2=\frac{\pi}{6}$:

$$\int_1^{\sqrt{3}}\frac{dx}{x^4\cdot\sqrt{4-x^2}}=-\int_{\frac{\pi}{3}}^{\frac{\pi}{6}}\frac{2\sin tdt}{2^4\cdot\cos^4 t\cdot 2\sin t}=\frac{1}{16}\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{dt}{\cos^4 t}=$$

$$=\frac{1}{16}\cdot\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(1+\mathrm{tg}^2 t\right)\frac{dt}{\cos^2 t}=\frac{1}{16}\left(\mathrm{tg}\, t+\frac{1}{3}\mathrm{tg}^3 t\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{11\sqrt{3}}{108}.$$

Example 14.5. Calculate definite integral: $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{dx}{\sin x\cos^3 x}$.

The solution. Let's make a variable substitution: $t=\mathrm{tg}\, x$; $dx=\frac{dt}{\cos^2 t}$; $t_1=1;t_2=\sqrt{3}$.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{dx}{\sin x\cos^3 x}=\int_1^{\sqrt{3}}\frac{1+t^2}{t}dt=\left(\ln|t|+\frac{t^2}{2}\right)\Bigg|_1^{\sqrt{3}}=\ln\sqrt{3}+\frac{3}{2}-\frac{1}{2}=\frac{1}{2}\ln 3+1.$$

Example 14.6. Prove that the given integral can be found using the formulas (integral with symmetric limits of integration):

$$\int_{-a}^{a} f(x)dx=\begin{cases}2\int_0^a f(x)dx,\ if\ \ f(x)-\text{odd function};\\ 0,\ if\ \ f(x)-\text{even function}.\end{cases}$$

Proof. Let us present the required integral $\int_{-a}^{a} f(x)dx$ in the form:

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx .$$

Next, in the first integral of the right-hand side of this relation, we will replace the variable $x = -t$ ($dx = -dt$). As a result, we get:

$$\int_{-a}^{a} f(x)dx = -\int_{a}^{0} f(-t)dt + \int_{0}^{a} f(x)dx \Rightarrow \int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-t)dt + \int_{0}^{a} f(x)dx,$$

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-t)dt + \int_{0}^{a} f(x)dx .$$

In the first integral on the right, we denote the variable of integration again by x, then

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(-x) + f(x)]dx .$$

It is obvious that the integrand function in the right part is equal to zero if the function $f(x)$ is odd and equal to $2 \cdot f(x)$ when the function $f(x)$ is even.

Found formulas are ***very useful***. For example, ***without calculations*** of the corresponding definite integrals, it is immediately possible to establish that

$$\int_{-a}^{a} x^5 \cdot e^{x^2} \cdot dx = 0 ; \quad \int_{-\pi}^{\pi} \sin^3 x \cdot \cos^2 x \cdot dx = 0 .$$

14.3. Integration by parts in a definite integral

Let's write down the formula for integration by parts in the definite integral:

$$\int_{a}^{b} u(x) \cdot v'(x)dx = u(x) \cdot v(x)\Big|_a^b - \int_{a}^{b} v(x) \cdot u'(x)dx , \qquad (14.3)$$

where $u(x)v(x)\Big|_a^b = u(b)v(b) - u(a)v(a)$.

Example 14.7. Calculate definite integral: $\int_{1}^{e} x^{\frac{3}{2}} \ln x dx$.

The solution. Let's put: $u = \ln x$, $du = \frac{dx}{x}$, $dv = x^{\frac{3}{2}}dx \Rightarrow v = \int x^{\frac{3}{2}}dx = \frac{2}{5}x^{\frac{5}{2}}$.

After substitution, we will have:

$$\int_1^e x^{\frac{3}{2}} \ln x dx = \frac{2}{5} x^{\frac{5}{2}} \cdot \ln x \Big|_1^e - \frac{2}{5} \int_1^e x^{\frac{3}{2}} dx = \frac{2}{5} e^{\frac{5}{2}} - \frac{4}{25}\left(e^{\frac{5}{2}} - 1 \right) = \frac{2}{25}\left(3e^{\frac{5}{2}} + 2 \right).$$

Example 14.8. Calculate definite integral: $J = \int_{-\pi/3}^{\pi/3} \frac{x \sin x}{\cos^2 x} dx$.

The solution. The integrand function is even, therefore

$$J = \int_{-\pi/3}^{\pi/3} \frac{x \sin x}{\cos^2 x} dx = 2 \int_0^{\pi/3} \frac{x \sin x}{\cos^2 x} dx .$$

Let's make a variable substitution:

$$u = x , \quad dv = \frac{\sin x}{\cos^2 x} dx , \quad du = dx , \quad v = \frac{1}{\cos x} .$$

Integrating by parts, we get:

$$\frac{J}{2} = \int_0^{\pi/3} \frac{x \sin x}{\cos^2 x} dx = \frac{x}{\cos x}\Bigg|_0^{\pi/3} - \int_0^{\pi/3} \frac{dx}{\cos x} = \frac{2\pi}{3} - \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2} \right)\Bigg|_0^{\pi/3} = \frac{2\pi}{3} - \ln \operatorname{tg}\left(\frac{5\pi}{12} \right).$$

Therefore, the required integral is equal to:

$$J = \frac{4\pi}{3} - 2 \ln \operatorname{tg}\left(\frac{5\pi}{12} \right).$$

Example 14.9. Calculate definite integral: $J = \int_{-1}^{1} x^5 \operatorname{arctg} x dx$.

The solution. The integrand function is even, therefore

$$J = \int_{-1}^{1} x^5 \operatorname{arctg} x dx = 2\int_0^1 x^5 \operatorname{arctg} x dx .$$

Let's make a variable substitution:

$$u = \operatorname{arctg} x , \quad dv = x^5 dx , \quad du = \frac{dx}{x^2 + 1} , \quad v = \frac{x^6}{6} .$$

Integrating by parts, we get:

$$\frac{J}{2} = \int_0^1 x^5 \operatorname{arctg} x dx = \frac{x^6 \operatorname{arctg} x}{6}\Bigg|_0^1 - \frac{1}{6}\int_0^1 \frac{x^6 dx}{x^2 + 1} = \frac{\pi}{24} - \frac{1}{6}\int_0^1 \left(x^4 - x^2 + 1 - \frac{1}{x^2 + 1} \right) dx =$$

$$= \frac{\pi}{24} - \frac{1}{6}\left(\frac{x^5}{5} - \frac{x^3}{3} + x - \operatorname{arctg} x \right)\Bigg|_0^1 = \frac{\pi}{24} - \frac{13}{90} + \frac{\pi}{24} = \frac{\pi}{12} - \frac{13}{90} .$$

Therefore, the required integral is equal to:

$$J = 2\left(\frac{\pi}{12} - \frac{13}{90}\right) = \frac{\pi}{6} - \frac{13}{45}.$$

Example 14.10. Calculate definite integral: $\int_0^1 x\ln(2-x)dx$.

The solution. Let's calculate the given integral by parts:

$$\int_0^1 x\ln(2-x)dx = \left| \begin{matrix} u = \ln(2-x); du = \dfrac{dx}{x-2}; \\ dv = xdx; v = \dfrac{x^2}{2} \end{matrix} \right| = \frac{x^2}{2}\cdot\ln(2-x)\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{x-2}dx =$$

$$= -\frac{1}{2}\int_0^1 \frac{x^2}{x-2}dx = -\frac{1}{2}\int_0^1 \frac{(x^2-4x+4)+(4x-8)+4}{x-2}dx = -\frac{1}{2}\int_0^1 (x-2)dx - 2\int_0^1 dx - 2\int_0^1 \frac{dx}{(x-2)} =$$

$$= \left(-2x - 2\ln|x-2| - \frac{1}{4}(x-2)^2\right)\Bigg|_0^1 = \ln 4 - \frac{5}{4}.$$

Example 14.11. Calculate the limit:

$$\lim_{x\to 0}\frac{\int_{2x}^{\sin 2x}(e^{t^2}-1)dt}{\int_{x}^{\sin x}\ln(1+t^2)dt}.$$

The solution. Since we have classical uncertainty (0/0), we will use Lhospital's rule

$$\lim_{x\to 0}\frac{\int_{2x}^{\sin 2x}(e^{t^2}-1)dt}{\int_{x}^{\sin x}\ln(1+t^2)dt} = \left\{\frac{0}{0}\right\} = \lim_{x\to 0}\frac{\left(\int_{2x}^{\sin 2x}(e^{t^2}-1)dt\right)'}{\left(\int_{x}^{\sin x}\ln(1+t^2)dt\right)'} =$$

$$= \lim_{x\to 0}\frac{(e^{\sin^2 2x}-1)\cdot 2\cdot\cos 2x - (e^{4x^2}-1)\cdot 2}{\ln(1+\sin^2 x)\cdot\cos x - \ln(1+x^2)\cdot 1} = |Taylor's\ formula| = \lim_{x\to 0}\frac{-\dfrac{80x^4}{3}+o(x^4)}{-\dfrac{5x^4}{6}+o(x^4)} = 32.$$

Tasks for classroom and independent work

I. Calculate the given definite integrals:

14.1. $\int\limits_1^2 \frac{e^{1/x^2}dx}{x^3}$.

14.2. $\int\limits_1^e \frac{dx}{x(1+\ln^2 x)}$.

14.3. $\int\limits_{-1}^1 \frac{xdx}{\sqrt{5-4x}}$.

14.4. $\int\limits_1^9 x\sqrt[3]{1-x}\,dx$.

14.5. $\int\limits_2^{4/\sqrt{3}} \frac{\sqrt{x^2-4}}{x}dx$.

14.6. $\int\limits_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}dx$.

14.7. $\int\limits_{\frac{\sqrt{2}}{2}}^1 \frac{\sqrt{1-x^2}}{x^2}dx$.

14.8. $\int\limits_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx$.

14.9. $\int\limits_0^{\pi/2} \frac{dx}{3+2\cos x}$.

14.10. $\int\limits_0^{2\pi} \frac{dx}{5-3\cos x}$.

14.11. $\int\limits_0^{3/4} \frac{dx}{(x+1)\sqrt{1+x^2}}$.

14.12. $\int\limits_{-1}^1 \frac{x^2+1}{x^4+1}dx$.

14.13. $\int\limits_{0,5}^2 \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$.

14.14. $\int\limits_0^{\sqrt{2}/2} \sqrt{\frac{1+x}{1-x}}dx$.

14.15. $\int\limits_0^1 x^2e^{3x}dx$.

14.16. $\int\limits_0^1 x^7 \operatorname{arctg} x\,dx$.

14.17. $\int\limits_0^1 x^2 \arcsin x\,dx$.

14.18. $\int\limits_0^1 x^2 \cos x\,dx$.

14.19. $\int\limits_1^e (x\ln x)^2 dx$.

14.20. $\int\limits_0^{\pi/4} e^{3x}\sin 4x\,dx$.

14.21. $\int\limits_0^1 \frac{\sqrt[3]{x}+\sqrt[6]{x}}{(\sqrt{x}+1)\sqrt[3]{x^2}}dx$.

14.22. $\int\limits_1^{e^{\sqrt{\pi}}} \frac{\ln x\cdot\sin(\ln^2 x)}{x}dx$.

14.23. $\int\limits_0^{2\operatorname{arctg}\frac{1}{2}} \frac{\sin x+\cos x+1}{(3+5\cos x)\cos x}dx$.

14.24. $\int\limits_{-1}^1 \frac{dx}{(1+x^2)^2}$.

II. Calculate the limits:

14.25. $\lim\limits_{x\to 0}\dfrac{\int\limits_{x}^{\operatorname{arctg} 3x}\ln(1+t^2)dt}{\int\limits_{x}^{\operatorname{tg} 2x}(e^{t^2}-1)dt}$. **14.26.** $\lim\limits_{x\to 0}\dfrac{\int\limits_{\arcsin(2x)}^{\operatorname{arctg}(3x)}\sin(t^2)dt}{\int\limits_{\sin x}^{\operatorname{tg}(2x)}\operatorname{tg}(t^2)dt}$.

III. Find definite integrals in symmetric limits:

14.27. $\int\limits_{-1}^{1} xe^{\cos x}dx$. **14.28.** $\int\limits_{-\pi/4}^{\pi/4} x^{10}\sin^9 x dx$. **14.29.** $\int\limits_{-\pi}^{\pi}\dfrac{xdx}{3+2\cos x}$.

14.30. $\int\limits_{-1}^{1} x\sqrt{1+x^6}dx$.

IV. Using definite integrals, find the limits of such sums:

14.31. $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\left(\sin\dfrac{\pi}{n}+\sin\dfrac{2\pi}{n}+\sin\dfrac{3\pi}{n}+\cdots+\sin\dfrac{\pi(n-1)}{n}\right)\right)$.

14.32. $\lim\limits_{n\to\infty}\left(\dfrac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}\right)$, $(p>0)$.

14.33. $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\left(\sqrt{1+\dfrac{1}{n}}+\sqrt{1+\dfrac{2}{n}}+\sqrt{1+\dfrac{3}{n}}+\cdots+\sqrt{1+\dfrac{n}{n}}\right)\right)$.

14.34. $\lim\limits_{n\to\infty}\left(\dfrac{\sum\limits_{k=1}^{n}\sqrt{(nx+k)(nx+k+1)}}{n^2}\right)$, $(x>0)$.

14.35. $\lim\limits_{n\to\infty}\left(\dfrac{2^{\frac{1}{n}}}{n+1}+\dfrac{2^{\frac{2}{n}}}{n+\frac{1}{2}}+\dfrac{2^{\frac{3}{n}}}{n+\frac{1}{3}}+\cdots+\dfrac{2^{\frac{n}{n}}}{n+\frac{1}{n}}\right)$

Answers and instructions

14.1. $\dfrac{e-\sqrt[4]{e}}{2}$. **14.2.** $\dfrac{\pi}{4}$. **14.3.** $\dfrac{1}{6}$. **14.4.** $-66\dfrac{6}{7}$. **14.5.** $\dfrac{1}{3}(2\sqrt{3}-\pi)$.

14.6. $\sqrt{2}-\dfrac{2}{\sqrt{3}}+\ln\dfrac{2+\sqrt{3}}{1+\sqrt{2}}$. **14.7.** $1-\dfrac{\pi}{4}$. **14.8.** $4-\pi$. **14.9.** $\dfrac{2}{\sqrt{5}}arctg\dfrac{1}{\sqrt{5}}$.

14.10. $\dfrac{\pi}{2}$. **14.11.** $\dfrac{1}{\sqrt{2}}\ln\dfrac{9+4\sqrt{2}}{7}$. **14.12.** $\dfrac{\pi}{\sqrt{2}}$.

14.13. $\frac{3}{2}e^{5/2}$.   **14.14.** $\frac{\pi}{4}-\frac{\sqrt{2}}{2}+1$.   **14.15.** $\frac{5e^3-2}{27}$.

4.16. $\frac{19}{210}$.   **14.17.** $\frac{\pi}{6}-\frac{2}{9}$.   **14.18.** 4π .

14.19. $\frac{5e^3-2}{27}$.   **14.20.** $\frac{4}{25}(e^{3\pi/4}+1)$.   **14.21.** $6-\frac{2\pi}{\sqrt{3}}$.

14.22. 1.   **14.23.** $\frac{1}{6}\ln\left(\frac{135}{16}\right)$.   **14.24.** $\frac{\pi}{4}+\frac{1}{2}$.

14.25. $\frac{26}{7}$.   **14.26.** $\frac{19}{7}$.   **14.27.** 0.

14.28. 0.   **14.29.** 0.   **14.30.** 0.

14.31. $\frac{2}{\pi}$.   **14.32**. $\frac{1}{p+1}$.   **14.33.** $\frac{2}{3}(2\sqrt{2}-1)$.

14.34. $x+\frac{1}{2}$.   **14.35**. $\frac{1}{\ln 2}$.

PRACTICAL LESSON 15

Application of the definite integral in geometric and physical problems (beginning)

15.1. Calculation of areas of flat figures

15.1.1. Calculation of the areas of flat figures in rectangular ones Cartesian coordinates

We will briefly state the relevant theoretical provisions regarding the educational material of this practical session.

A. Let the function $f(x) \ge 0$ be given in Cartesian coordinates OXY on the segment $[a;b]$. Then the ***area of the curvilinear trapezoid***, which is bounded by the curve $y = f(x)$, the axis OX and the straight lines $x = a$, $x = b$, is equal to

$$S = \int_a^b f(x)dx. \tag{15.1}$$

If the function $f(x) \le 0$ is defined on $[a;b]$, then the definite integral is negative: $\int_a^b f(x)dx \le 0$. But in absolute terms, it is equal to the area S of the corresponding curvilinear trapezoid aA_2B_2b:

$$-S = \int_a^b f(x)dx. \tag{15.2}$$

Example 15.1. Calculate the area of the figure bounded by the curves $y(x) = \sin^3 x$ and $y = 0$ on a segment $x \in [0, \pi]$.

The solution. Draw the figure yourself. We calculate the area using formula (15.1):

$$S = \int_0^\pi \sin^3 x dx = -\int_0^\pi \sin^2 x d\cos x = -\int_0^\pi (1 - \cos^2 x) d\cos x = -\left(\cos x - \frac{1}{3}\cos^3 x\right)\Bigg|_0^\pi = \frac{4}{3}.$$

B. If it is necessary to calculate the area of the region $A_1A_2B_2B_1$ (Fig. 15.1), which is limited by curves $y = f_1(x)$, $y = f_2(x)$, and straight lines $x = a$, $x = b$, then under the condition that $f_2(x) \ge f_1(x)$, the area is calculated according to the following formula:

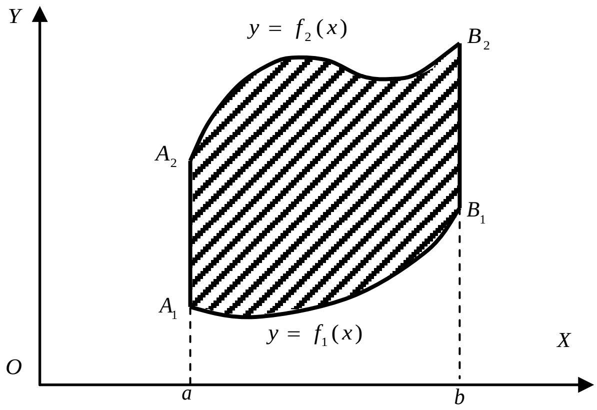

Fig. 15.1

$$S = \int_a^b f_2(x)dx - \int_a^b f_1(x)dx = \int_a^b [f_2(x) - f_1(x)]dx\,. \qquad (15.3)$$

If the flat figure has a more complex shape (Fig. 15.2), then it must be divided into a finite sum (difference) of curvilinear trapezoids by straight lines, $x = a$, $x = b$, $x = c$, $x = d$, which are parallel to the axis OY. Then the area of the figure will be equal to the algebraic sum of the areas of the formed regions $(S_1), (S_2), (S_3)$, i.e. the area S of the complex figure will be calculated by the formula

$$S = S_1 + S_2 + S_3\,.$$

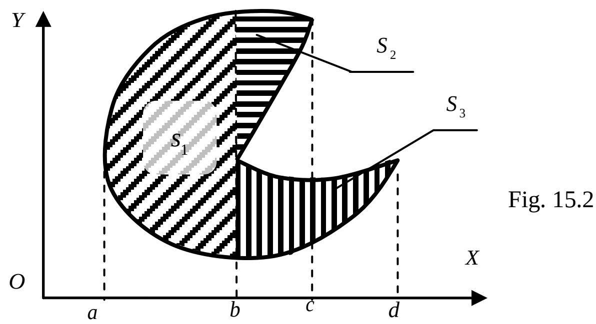

Fig. 15.2

Example 15.2. Calculate the area of the figure bounded by the curves $y = \sqrt{x}$ and $y = x^2$.

The solution. Let's draw a figure, the area of which must be found (Fig. 15.3). We find the points of intersection of the curves: $\sqrt{x} = x^2 \Rightarrow x = x^4 \Rightarrow x_1 = 0;\ x_2 = 1$. Then we calculate the area using formula (15.3):

$$S = \int_0^1 \sqrt{x}dx - \int_0^1 x^2 dx = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \frac{2}{3} x^{\frac{3}{2}}\Bigg|_0^1 - \frac{1}{3} x^3\Big|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

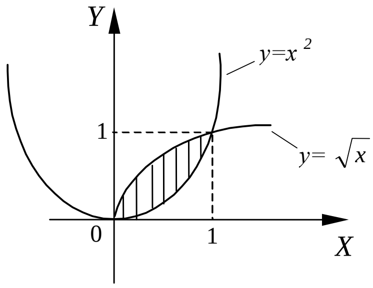

Fig. 15.3

Example 15.3. Find the area of the figure bounded by a straight line $y = x$ and a parabola $y = 2 - x^2$ (Fig. 15.4).

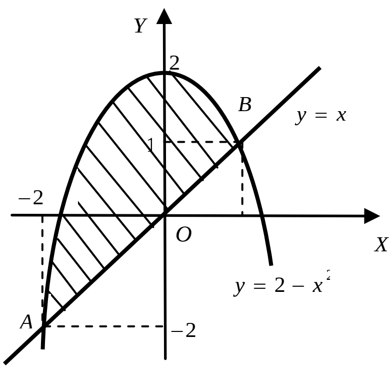

Fig. 15.4

The solution. Let's find the abscissas of the points A and B the intersection of these lines. They will be the limits of integration:

$$\begin{cases} y = 2 - x^2 \\ y = x \end{cases} \Rightarrow x_1 = -2; \quad x_2 = 1$$

Using formula (15.3), we find the area of the given figure:

$$S = \int_{-2}^{1} \left[\left(2 - x^2\right) - x\right] dx = \left(2x - \frac{x^3}{3} - \frac{x^2}{2}\right)\Bigg|_{-2}^{1} = \frac{9}{2}.$$

15.1.2. Calculation of areas of flat figures bounded by curves, which are given parametrically

Let's calculate the area of the curved trapezoid (Fig. 15.5) if the curve $y = f(x)$ is given by parametric equations (15.4), where the parameter t changes on the segment: $\alpha \le t \le \beta$,

$$\begin{cases} x(t) = \varphi(t) \\ y(t) = \psi(t) \end{cases}, \tag{15.4}$$

Fig. 15.5

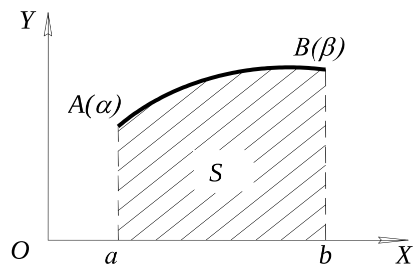

Fig. 15.5

After replacing the variable in integral (15.1) ($x = \varphi(t)$, $dx = \varphi'(t)dt$) based on equation (15.4), we get: $y = f(x) = f[\varphi(t)] = \psi(t)$. Then the integral (15.1) takes the following form:

$$S = \int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = \int_\alpha^\beta \psi(t)\varphi'(t)dt. \tag{15.5}$$

Let's consider two examples.

Example 15.4. Calculate the area of the figure bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The solution. Let's calculate the area of the upper half of the ellipse, and then double the result. The parametric equation of the ellipse has the form $x = a\cos t$, $y = b\sin t$ $(0 \le t \le 2\pi)$. For the upper half of the ellipse, the parameter t varies from π to 0 and $dx = -a\sin t dt$. **Importantly!** To get the correct result, it is necessary that the variable x increases. That's why the limits for t are chosen exactly like this. Using formula (15.5), we obtain:

$$S = 2\int_\pi^0 b\sin t(-a\sin t)dt = -2ab\int_\pi^0 \sin^2 t dt = -2ab\int_\pi^0 \frac{1-\cos 2t}{2}dt = -ab\left[t - \frac{\sin 2t}{2}\right]\Bigg|_\pi^0 = \pi ab.$$

This problem can be solved without going to the parametric form of the ellipse equation. Let's show it. We express y through x for the part of the ellipse located in the first quarter of the coordinate plane: $y = \frac{b}{a}\sqrt{a^2 - x^2}$, $0 \le x \le a$. Let's find the fourth part of the area of the ellipse and then multiply it by four times to calculate the total area of the ellipse:

$$S_{1/4} = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \left| \begin{matrix} x = a\sin t;\; dx = a\cos t dt;\; t_1 = 0; \\ \sqrt{a^2 - x^2} = a\cos t;\; t_2 = \pi/2 \end{matrix} \right| = \frac{b}{a}\int_0^{\pi/2} a^2\cos^2 t dt =$$

$$= \frac{ab}{2}\int_0^{\pi/2}(1 + \cos 2t)dt = \frac{\pi ab}{4}.$$

Therefore, the total area of the ellipse is equal to $S = \pi ab$. This formula is very useful further on.

Example 15.5. Calculate the area of the loops of the curve given by the equation: $(x + y)^3 = xy$.

The solution. Draw the figure yourself. Let's convert the given equation to the parametric form using substitution: $y = xt$. Then the equation of the curve will take the following form:

$$\begin{cases} y = \dfrac{t^2}{(1+t)^3}; \\ x = \dfrac{t}{(1+t)^3}. \end{cases}$$

Let's set the limits of integration. The curve forms a loop when the parameter runs through the values in the interval $t \in [0, \infty)$. However, it should be noted here that before calculating the corresponding integral, it is necessary to set the value of the parameter t, at which the variable x acquires the largest value. This value is equal to $t = \frac{1}{2}$. The resulting integral is called an improper integral of the first kind. We will get acquainted with such integrals in Practical Lesson No 17. Therefore, it is necessary to calculate the following integral:

$$S = \int_{\infty}^{\frac{1}{2}} y(t)dx(t) - \int_0^{\frac{1}{2}} y(t)dx(t) = -\int_{\frac{1}{2}}^{\infty} y(t)dx(t) - \int_0^{\frac{1}{2}} y(t)dx(t) =$$

$$= -\int_0^{\infty} \frac{t^2}{(1+t)^3}d\left(\frac{t}{(1+t)^3}\right) = -\int_0^{\infty}\frac{t^2(1-2t)}{(1+t)^7}dt = \frac{1}{60}.$$

Intermediate calculations in the integration process are omitted here.

Importantly! We draw your attention to the limits of the corresponding integrals, the content of which can be understood if the loop diagram is constructed and analyzed in detail.

15.1.3. Calculation of areas of flat figures bounded by curves,which are specified in the polar coordinate system

Before studying this educational material, you need to repeat the educational material about the ***polar coordinate system***.

Definition. *The figure OAB, which is bounded by a curve given by a continuous function $\rho = \rho(\varphi)$ and* ***rays*** *with* ***polar angles*** *$\varphi = \alpha$ and $\varphi = \beta$, is called a* ***curvilinear sector*** *(Fig. 15.6).*

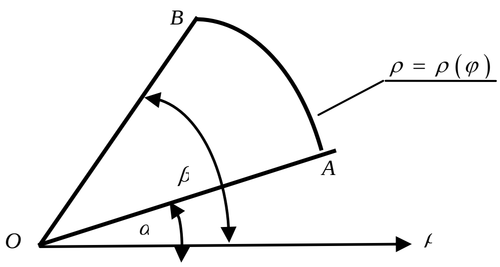

Fig. 15.6

The ***area*** of ***the curved sector*** is calculated by the formula

$$S = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2(\varphi)d\varphi \,. \qquad (15.6)$$

Example 15.6. Calculate the area bounded by the "***three-petal rose***" $\rho(\varphi) = a \cdot \cos 3\varphi$ (Fig. 15.7).

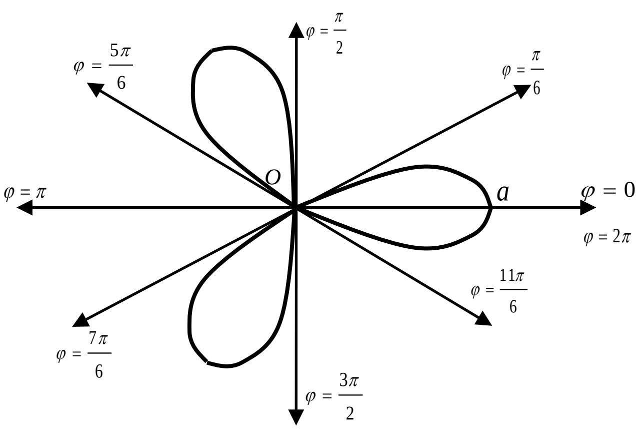

Fig. 15.7

Fig. 15.7

The solution. Let’s use the symmetry of this figure. Let’s find the area of half of the first petal of the “***rose***”, which corresponds to the angles from the line segment $0 \le \varphi \le \frac{\pi}{6}$, and multiply it by six. Therefore, according to formula (15.6), we have:

$$S = 6 \cdot \frac{1}{2} \int_0^{\pi/6} (a\cos 3\varphi)^2 d\varphi = 3a^2 \int_0^{\pi/6} \frac{1+\cos 6\varphi}{2} d\varphi = \frac{3}{2} a^2 \left(\varphi + \frac{\sin 6\varphi}{6} \right) \Bigg|_0^{\pi/6} = \frac{\pi a^2}{4}.$$

Example 15.7. Calculate the area of the figure bounded by ***Bernoulli’s lemniscate*** (Fig. 15.8): $\rho(\varphi) = a \cdot \sqrt{\cos 2\varphi}$; $0 \le \varphi \le \frac{\pi}{4}$; $\frac{3\pi}{4} \le \varphi \le \frac{5\pi}{4}$.

The solution. Let’s plot this curve and use the symmetry of this figure. Let’s find the area of the fourth part of the required area using formula (15.6).

$$\frac{1}{4} S = \frac{1}{2} \int_0^{\pi/4} \rho^2(\varphi) d\varphi = \frac{1}{2} a^2 \int_0^{\pi/4} \cos 2\varphi d\varphi = \frac{a^2}{2} \frac{\sin 2\varphi}{2} \Bigg|_0^{\pi/4} = \frac{a^2}{4} \Rightarrow S = a^2.$$

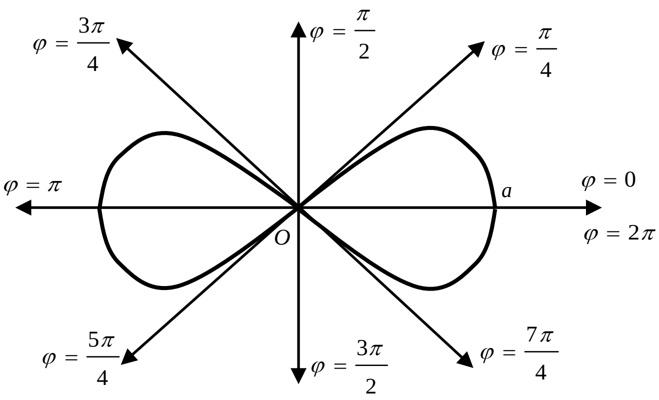

Fig. 15.8

Example 15.8. Find the area of one loop of a figure bounded by a curve: $(x^2 + y^2)^2 = 4x^2 y$. Let’s draw of the specified curve (Fig. 15.9) in the polar coordinate system. The equation of the curve in the polar coordinate system is given below.

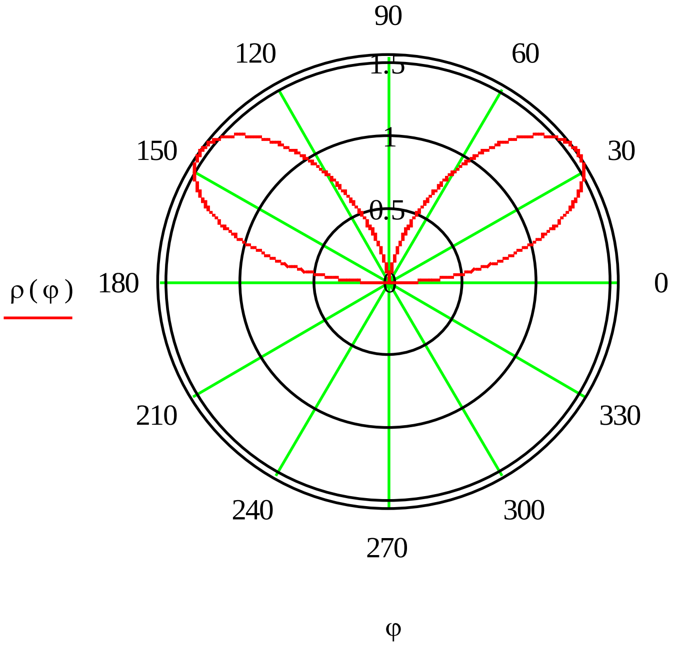

Fig. 15.9

The solution. Before solving this problem, you need to build the specified curve yourself. Let's move to the polar coordinate system, then the equation of the given curve takes the form: $\rho = 4\sin\varphi \cdot \cos^2\varphi$. To obtain the area of a figure bounded by one curve loop (Fig. 15.9), formula (15.6) should be used and integrated within $\varphi \in [0;\pi/2]$:

$$S = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(4\sin\varphi\cos^2\varphi\right)^2 d\varphi = 8\int_0^{\frac{\pi}{2}} \sin^2\varphi\cos^4\varphi d\varphi = \int_0^{\frac{\pi}{2}} \sin^2 2\varphi(1+\cos 2\varphi)d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 2\varphi d\varphi + \int_0^{\frac{\pi}{2}} \sin^2 2\varphi\cos 2\varphi d\varphi = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2\varphi)d\varphi + \underbrace{\frac{1}{3}\sin^3 2\varphi\Big|_0^{\frac{\pi}{2}}}_{=0} = \frac{\pi}{4}.$$

15.2. Calculation of the length of the arc of the curve

15.2.1. Calculation of the arc length of a plane curve given by the equation in parametric form

Definition 15.1. *Let the functions* $\varphi(t)$ *and* $\psi(t)$ *be given and continuous on the set* $\{t\}$. *Let's say that two equations*

$$x = \varphi(t), \quad y = \psi(t), \quad t \in [\alpha, \beta], \tag{15.7}$$

*set a **simple flat curve*** L *if different points of the curve* L *correspond to different values of the parameter* t *from the segment* $[\alpha, \beta]$.

From this definition it follows that a simple plane curve L has no self-intersection points.

Definition 15.2. *A curve L described by equations (15.7) is called a **parametrically defined curve** if there exists such a system of segments $\{[t_{i-1}, t_i]\}$ that divide the set $\{t\}$ so that for values t from each given elementary segment of this system, equations (15.7) define a **simple curve**.*

Now let's introduce the concept of the arc ***length of a parametric curve***.

Let the curve L be given by the parametric equation (15.7), where the parameter t changes over the segment $[\alpha, \beta]$. Let $T-$ be an arbitrary division of the segment $[\alpha, \beta]$ by points

$\alpha = t_0 < t_1 < \cdots < t_{n-1} < t_n = \beta$.

Next, we will call it $T-$ the division of the segment $[\alpha, \beta]$. Let us denote by letters $M_0, M_1, M_2, \cdots, M_n$ the points of the curve L corresponding to this division.

Definition 15.3. *We will call the broken line $M_0 M_1 M_2 \cdots M_n$ that was formed in this case the **broken line inscribed in the curve** L and corresponding to the given $T-$ division of the segment $[\alpha, \beta]$.*

Since the length l_i of the elementary link $M_{i-1}M_i$ of this broken one is equal to

$$l_i = \sqrt{[\varphi(t_i) - \varphi(t_{i-1})]^2 + [\psi(t_i) - \psi(t_{i-1})]^2} ,$$

then the length $\overline{l}(t_i)$ of this broken line will be equal to:

$$\overline{l}(t_i) = \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} \sqrt{[\varphi(t_i) - \varphi(t_{i-1})]^2 + [\psi(t_i) - \psi(t_{i-1})]^2} = \sum_{i=1}^{n} \sqrt{[\Delta\varphi(t_i)]^2 + [\Delta\psi(t_i)]^2} . \quad (15.8)$$

Definition 15.4. *If the set $\{\overline{l}(t_i)\}$ of lengths of broken lines inscribed in the curve L, which correspond to arbitrary T-partitions of the segment $[\alpha, \beta]$, is **bounded**, then the curve L is said to **straighten**, and the **exact upper edge of the set** is called the **length** of **the arc of the curve**.*

Remarks 15.1. From the definition of a parametrically given curve L and the definition of the length l of an arc of such a curve, it follows that the length l is always positive, $l > 0$.

Remark 15.2. There are ***non-straightening curves***. Examples of such curves can be found in [1, 2]. We will consider only those curves that straighten.

Let us formulate ***sufficient conditions*** for the curve to be such that it ***straightens***. Let's write down the formula for calculating the arc length of a parametrically specified curve L.

Theorem (*on sufficient conditions for straightening a curve*). *If the functions $\varphi(t)$ and $\psi(t)$ on the segment $[\alpha, \beta]$ have **continuous derivatives**, then* *the curve L defined by the parametric equations (15.7) is the one that* ***straightens****, and the length l of its arc is calculated by the formula:*

$$l=\int_{\alpha}^{\beta}\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt\,. \tag{15.9}$$

Remarks 15.3. Formulas (15.8) and (15.9) resemble the well-known Pythagorean Theorem for finite segments. They represent a certain generalization of it for the case of infinitesimally small increments $\Delta\varphi(t)$ and $\Delta\psi(t)$ functions (15.7), if the parameter t is given an increment Δt. However, this is possible only under the condition that the curve L defined by parametric equations (15.7) is the one that ***straightens***.

Remark 15.4. Formula (15.9) remains true for such curves that are intersected by vertical straight lines at more than one point, if only both derivatives were ***continuous*** at all points of the curve: $\varphi'(t)$ and $\psi'(t)$. Such a curve under the condition that $\varphi'(t)\neq 0$, $\psi'(t)\neq 0$, $\forall t\in[\alpha;\beta]$, is called ***smooth***.

Now let the ***spatial curve*** be given in parametric form:

$$x=\varphi(t),\;\; y=\psi(t),\;\; z=\chi(t),\;\; (\alpha\le t\le\beta)\,.$$

If the functions $x=\varphi(t)$, $y=\psi(t)$, $z=\chi(t)$ are ***continuous*** and have ***continuous derivatives*** on the segment $[\alpha,\beta]$, then the length of the curve is calculated by the formula:

$$l=\int_{\alpha}^{\beta}\sqrt{\left[\varphi'(t)\right]^2+\left[\psi'(t)\right]^2+\left[\chi'(t)\right]^2}\,dt. \tag{15.10}$$

Example 15.9 (*Flat curve*). Calculate the arc length of the ***asteroid*** $x=a\cos^3 t$, $y=a\sin^3 t$ (Fig. 15.10).

The solution. Since the curve is symmetrical with respect to both coordinate axes, we will first calculate the length of $1/4$ the part contained in the first quarter. Let's find the corresponding derivatives:

$$\frac{dx}{dt}=-3a\cos^2 t\sin t\,;\quad \frac{dy}{dt}=3a\sin^2 t\cos t\,,\quad 0\le t\le\frac{\pi}{2}\,.$$

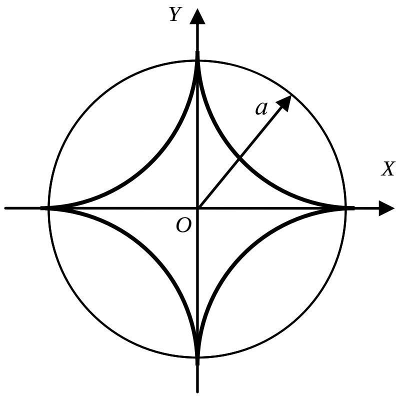

Fig. 15.10

Then by formula (15.10) we get:

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t}\,dt = 12a\int_0^{\frac{\pi}{2}} \sqrt{\cos^2 t\sin^2 t}\,dt =$$

$$= 12a\int_0^{\frac{\pi}{2}} \cos t\sin t\,dt = 12a\left.\frac{\sin^2 t}{2}\right|_0^{\frac{\pi}{2}} = 6a.$$

Example 15.10 (*spatial curve*). Calculate the length of the arc of the ***spiral line*** $x = a\cos t$, $y = a\sin t$, $z = amt$, when t changing from 0 to 2π.

The solution. From the given equations we find: $dx = -a\sin t dt$, $dy = a\cos t dt$; $dz = am dt$. Substitute into formula (15.10):

$$L = \int_0^{2\pi} \sqrt{a^2\sin^2 t + a^2\cos^2 t + a^2m^2}\,dt = a\int_0^{2\pi} \sqrt{1+m^2}\,dt = 2\pi a\sqrt{1+m^2}.$$

15.2.2. Calculation of the arc length of a plane curve given in a rectangular Cartesian coordinate system

Let a differentiable function $y = f(x)$ on a segment $[a,b]$ be given, the graph of which is a certain curve L on the plane XOY. In addition, we consider that the given curve is one that ***straightens***. Therefore, this curve can be specified in the parametric form (15.7), and besides $a = \varphi(\alpha)$ and $b = \varphi(\beta)$. Let's find the length l of the arc $\cup AB$ of this curve L, which is contained between the vertical lines $x = a$, $x = b$.

From formula (15.9), we write the ***differential*** for the elementary arc of the curve L as follows

$$|dl| = \sqrt{[\varphi'(t)]^2 + [\psi'(t)]^2}\,dt = \sqrt{[\varphi'(t)dt]^2 + [\psi'(t)dt]^2} = \sqrt{(dx)^2 + (dy)^2}\,.$$

The last formula can be transformed as follows:

$$|dl| = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + [\,f'(x)]^2}\,dx\,.$$

Let's integrate the left and right sides of the last equality in the range from a to b and get the formula for calculating the ***length*** l of ***the arc*** using the definite integral:

$$l = \int_a^b \sqrt{1 + [\,f'(x)]^2}\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx\,. \tag{15.11}$$

Example 15.11. Find the length of the chain line $y = \dfrac{e^x + e^{-x}}{2}$, $x \in [0,1]$.

The solution. Because, $y' = \dfrac{e^x - e^{-x}}{2}$, $0 \le x \le 1$, then by formula (15.11) we obtain:

$$l = \int_0^1 \sqrt{1 + \left(\frac{e^x - e^{-x}}{2}\right)^2}\,dx = \int_0^1 \sqrt{\frac{e^{2x} + 2 + e^{-2x}}{4}}\,dx = \int_0^1 \frac{e^x + e^{-x}}{2}\,dx = \left(\frac{e^x - e^{-x}}{2}\right)\Bigg|_0^1 = \frac{e - e^{-1}}{2} = \frac{e^2 - 1}{2e}\,.$$

Example 15.12. Find the length of the arc of the curve $y(x) = \ln(\sin x)$ on the interval $x \in \left[\dfrac{\pi}{3}, \dfrac{\pi}{2}\right]$.

The solution. Let's find the derivative of the given function: $y'(x) = \operatorname{ctg} x$. Substitute the found derivative into formula (15.11):

$$l = \int_{\pi/3}^{\pi/2} \sqrt{1 + \operatorname{ctg}^2 x}\,dx = \int_{\pi/3}^{\pi/2} \frac{dx}{\sin x} = \ln\left|\operatorname{tg}\frac{x}{2}\right|\Bigg|_{\pi/3}^{\pi/4} = 0 - \ln\left(\frac{\sqrt{3}}{3}\right) = \frac{\ln 3}{2}\,(linear\ units)$$

15.2.3. Calculation of the arc length of a plane curve, which is specified in polar coordinates

Let the equation of the curve be given in polar coordinates

$$\rho = f(\varphi)\,, \tag{15.12}$$

where ρ – is the polar radius; φ – the polar angle; $\varphi_1 \le \varphi \le \varphi_2$.

Let's write down the formula for calculating the arc length of a plane curve, which is given in polar coordinates:

$$L = \int_{\varphi_1}^{\varphi_2} \sqrt{\rho^2 + \rho'^2}\,d\varphi. \tag{15.13}$$

Example 15.13. Find the length of the cardioid $\rho(\varphi) = a(1+\cos\varphi)$.

The solution. If the polar angle φ varies within $0 \le \varphi \le \pi$, then when integrating within these limits, we get half of the required length (Fig. 15.11, $a = 2$).

Let's find the derivative ρ': $\rho' = -a\sin\varphi$. Now we use the formula (15.13):

$$L = 2\int_0^{\pi} \sqrt{a^2(1+\cos\varphi)^2 + a^2\sin^2\varphi}\, d\varphi = 2a\int_0^{\pi}\sqrt{2+2\cos\varphi}\, d\varphi =$$

$$= 4a\int_0^{\pi}\cos\frac{\varphi}{2}\, d\varphi = 8a\sin\frac{\varphi}{2}\Big|_0^{\pi} = 8a.$$

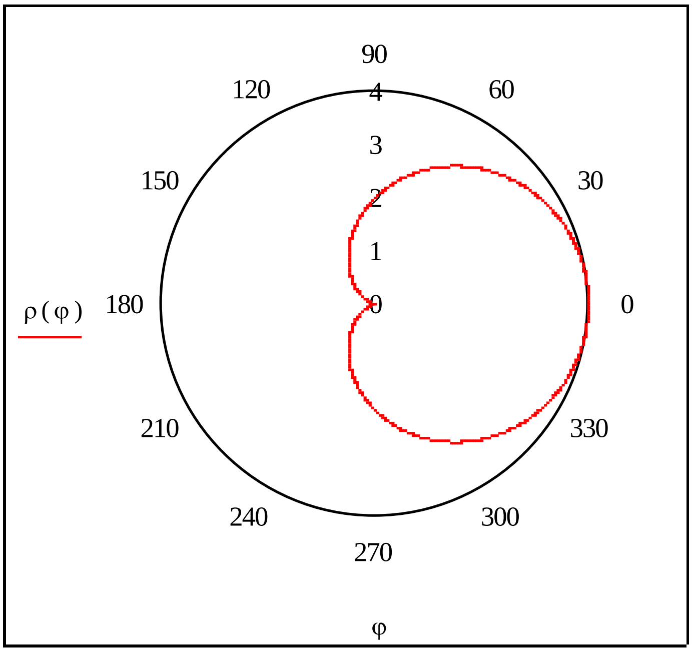

Fig. 15.11

Tasks for classroom and independent work

I. Find the areas of figures bounded by given lines:

15.1. $y = 2x - x^2,\ x + y = 0$. **15.2.** $x^2 + y^2 = 41,\ xy = 20,\ x \ge 0,\ y \ge 0$.

15.3. $y = (x-3)^2,\ y = 9 - x^3$. **15.4.** $x = \sqrt{2y};\ x = \sqrt{y+2}$.

15.5. $\rho = 2(1+\cos\varphi)$. **15.6.** $\rho = a\sin 3\varphi$.

15.7. $x = 2t - t^2,\ y = 2t^2 - t^3$. **15.8.** $D:\{x = a\cos^4 t,\ y = b\sin^4 t,\ x = 0,\ y = 0\}$.

15.9. Find the area of the figure bounded by the line $x^4 + y^4 = x^2 + y^2$.

15.10. Find the area of the figure bounded by the lines $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2ay$.

15.11. In what ratio does the parabola $y^2 = 2x$ divide the area of the circle $x^2 + y^2 = 8$?

II. Find the lengths of the arcs of the curves.

15.12. $x = \frac{1}{4}y^2 - \frac{1}{2}\ln y, 1 \le y \le e$. **15.13.** $y = \ln x, \sqrt{3} \le x \le \sqrt{8}$.

15.14. $x = e^t \cos t;\ y = e^t \sin t;\ 0 \le t \le \ln \pi$. **15.15.** $\rho = a \sin^3 \frac{\varphi}{3}$.

15.16. $\rho(\varphi) = a \sin^4 \frac{\varphi}{4}$.

15.17. $x = a(t - \sin t),\ y = a(1 - \cos t);\ 0 \le t \le 2\pi$.

15.18. Find the length of the arc of the curve $y = \sqrt{x - x^2} + \arcsin\sqrt{x}$.

15.19. Find the length of one arch of the cycloid:

$$x = a(t - \sin t);\ y = a(1 - \cos t);\ 0 \le t \le 2\pi.$$

15.20. Find the loop length of the plane curve $L: \left\{ x = t^2; y = t - \frac{1}{3}t^3 \right\}$.

15.21. Find the perimeter of the hole formed by the circles

$$x^2 + y^2 = 2ax \text{ i } x^2 + y^2 = 2by, (a > b > 0).$$

III. Solve various problems on the topic of calculating the areas of flat figures and the lengths of arcs of curves.

15.22. Find the area of a figure bounded by a line $x^4 + y^4 = x^2 + y^2$.

15.23. Calculate the area of the loops of the curve given by the equation: $(x + y)^4 = x^2 y$.

15.24. Find the area of a figure bounded by line $x = 12\cos t + 5\sin t;\ y = 5\cos t - 12\sin t$.

15.25. Find the area of a figure bounded by line $\rho = \dfrac{4}{\cos\left(\varphi - \frac{\pi}{6}\right)}$, $\varphi = \left[\frac{\pi}{6}; \frac{\pi}{3}\right]$.

15.26. Calculate the length of the line $\rho(\varphi) = a \operatorname{th}\left(\frac{\varphi}{2}\right), \varphi \in [0, 2\pi]$.

15.27. Calculate the length of the line $x = \cos^5 t;\ y = \sin^5 t;\ 0 \le t \le \frac{\pi}{2}$.

15.28. Calculate the arc length of the spatial curve

$$L: \left\{ x = a\sqrt{t}\cos t; y = a\sqrt{t}\sin t; z = at \right\},\ t \in [0;1].$$

15.29. Calculate the length of the line $L:\{x = a\cos^4 t,\ y = b\sin^4 t,\ a > 0,\ b > 0\}$.

15.30. Calculate the length of the line $\rho(\varphi) = \dfrac{r}{1+\cos\varphi}$, $|\varphi| \le \dfrac{\pi}{2}$.

Answers and instructions

15.1. 4,5. **15.2.** $41(\arcsin\frac{5}{\sqrt{41}} + \arcsin\frac{4}{\sqrt{41}}) - 40\ln\frac{5}{4}$.

15.3. $26\frac{5}{12}$. **15.4.** $8/3$. **15.5.** 6π. **15.6.** $\pi a^2/4$.

15.7. $8/15$. **15.8.** $\frac{ab}{6}$. **15.9.** $\pi\sqrt{2}$. **15.10.** $a^2(\pi/2-1)$.

15.11. $\frac{3\pi+2}{9\pi-2}$. **15.12.** $(e^2+1)/4$. **15.13.** $1+\frac{1}{2}\ln\frac{3}{2}$.

15.14. $\sqrt{2}(\pi-1)$. **15.15.** $3\pi a/2$. **15.16.** $16a/3,\ 0 \le \varphi \le 4\pi$.

15.17. $8a$. **15.18.** 2. **15.19.** $8a$. **15.20.** $4\sqrt{3}$.

15.21. $\pi a - 2(a-b)arctg\frac{a}{b}$. **15.22.** $\pi\sqrt{2}$. **15.23.** $\frac{1}{210}$.

15.24. 169π. **15.25.** $\frac{8\sqrt{3}}{3}$. **15.26.** $a(2\pi - \operatorname{th}\pi)$.

15.27. $\frac{5}{8}\left[2 - \frac{\ln(2-\sqrt{3})}{\sqrt{3}}\right]$. **15.28.** $5a/3$.

15.29. $1+\frac{\ln(1+\sqrt{2})}{\sqrt{2}}$. **15.30.** $r\left[\sqrt{2}+\ln(1+\sqrt{2})\right]$.

PRACTICAL LESSON 16

Application of the definite integral in geometric and physical problems (end).

16.1. Calculation of body volume

16.1.1. Calculation of the volume of the body by the areas of its cross sections

The volume V of a material body can be found if ***the cross-sectional areas*** S of this body are known in planes perpendicular to some axis, for example OX (Fig. 16.1), $S = S(x)$, $a \le x \le b$.

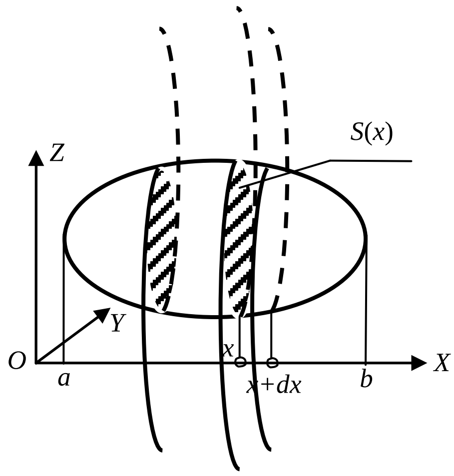

Fig.16.1

The formula for this volume is as follows:

$$V = \int_a^b S(x)dx, \qquad (16.1)$$

Example 16.1. Calculate the volume of the triaxial ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

The solution. Let's fix a certain value of the appliqué $z = z_0$ and write down the equation of the ellipse, which is formed when the given ellipsoid is cut by a plane $z = z_0$

$$\frac{x^2}{a^2\left(1 - \frac{z_0^2}{c^2}\right)} + \frac{y^2}{b^2\left(1 - \frac{z_0^2}{c^2}\right)} = 1.$$

It is known that the area of a figure bounded by an ellipse $\frac{x^2}{\overline{a}^2}+\frac{y^2}{\overline{b}^2}=1$ is calculated by the formula: $S=\pi\overline{a}\overline{b}$, where $\overline{a}=a\sqrt{\left(1-\frac{z_0^2}{c^2}\right)}$, $\overline{b}=b\sqrt{\left(1-\frac{z_0^2}{c^2}\right)}$.

We use formula (16.1) to calculate the volume of the ellipsoid, taking into account that z_0 was chosen arbitrarily and the index can be removed. Therefore, $S(z)=\pi ab\left(1-\frac{z^2}{c^2}\right)$, and the volume differential has the form $dV=S(z)dz$; then the volume V of the ellipsoid is equal to:

$$V=2\int_0^c \pi ab\left(1-\frac{z^2}{c^2}\right)dz=2\pi ab\left(z-\frac{z^3}{3c^2}\right)\Bigg|_0^c=\frac{4}{3}\pi abc\,.$$

If $a=b=c=R$ (that is, the ellipsoid turns into a sphere), then the volume of the latter is calculated according to the well-known formula: $V=\frac{4}{3}\pi R^3$.

Example 16.2. Calculate the volume of the body T bounded by the paraboloid and the cone $T:\left\{x^2+y^2=2-z;\ \sqrt{x^2+y^2}=z\right\}$.

The solution. Draw a body T yourself. This task is to apply formula (16.1) to calculate the volume V of a body based on its cross-sectional areas. Let's find the equation of the plane along which these two surfaces (paraboloid and cone) intersect:

$$z^2+z-2=0\Rightarrow z=1\,.$$

Thus, the body T is bounded below by a cone, and above by a paraboloid, and these surfaces intersect in the plane $z=1$. Note that in any section of the body T in a plane parallel to the plane $z=0$, we will have a circle of the corresponding variable radius: either $\rho=z$ (for a cone) or $\rho=\sqrt{2-z}$ (for a paraboloid). Therefore, using the considerations of the previous problem, we can write down the following integrals for calculating the body volume:

$$V=\pi\int_0^1 z^2dz+\pi\int_1^2(2-z)dz=\pi\left(\frac{1}{3}-\left(\frac{(2-z)^2}{2}\right)\Bigg|_1^2\right)=\frac{5\pi}{6}\,.$$

16.1.2. Volume of bodies of rotation

Let the curved trapezoid be bounded from above by the graph of the continuous function $y = f(x) \geq 0$, $a \leq x \leq b$. If this trapezoid is rotated around the axis, a spatial figure called a ***body of rotation*** will be formed (Fig. 16.2).

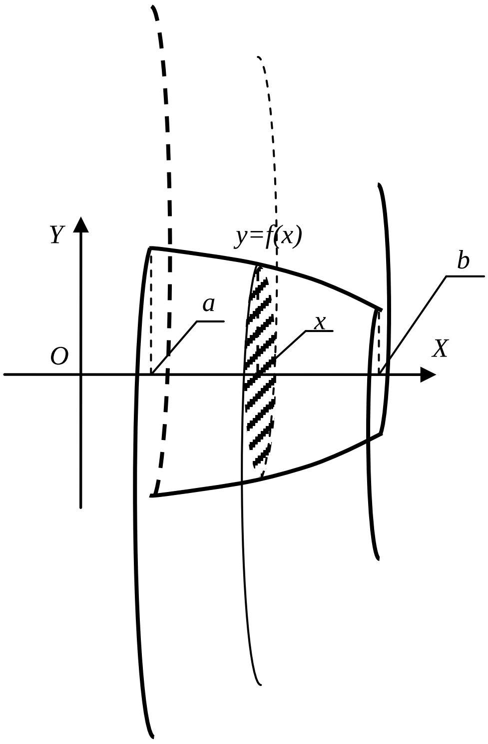

Fig. 16.2

Since the area of the parallel section of the figure at the point x is equal to $S(x) = \pi y^2 = \pi[f(x)]^2$, then, according to formula (16.1), the volume V_x of the body formed by the rotation of this trapezoid around the axis OX is calculated according to the formula:

$$V_x = \pi \int_a^b f^2(x)dx . \tag{16.2}$$

If a curved trapezoid is bounded by the graph of a continuous function $x = \varphi(y) \geq 0$ and straight lines $y = c$, $y = d$, $x = 0$, then the volume V_y of the body formed by the rotation of this trapezoid around the axis OY is found by the formula

$$V_y = \pi \int_c^d \varphi^2(y)dy . \tag{16.3}$$

Example 16.3. Find the volume of the body formed by the rotation of the parabola $y = x^2$ on the interval $1 \le x \le 2$ around:

a) the axis OX;

b) the axis OY.

The solution. According to formulas (16.2) and (16.3), respectively, we obtain:

$$V_x = \pi \int_1^2 x^4 dx = \pi \left. \frac{x^5}{5} \right|_1^2 = \frac{31\pi}{5}; \; V_y = \pi \int_1^4 y dy = \left. \frac{\pi y^2}{2} \right|_1^4 = \frac{15\pi}{2}.$$

16.1.3. Calculation of the surface area of rotation

Let the curve given by the continuous function $y = f(x) \ge 0$, $a \le x \le b$, rotate around the axis OX. Let us cross the surface of rotation with two infinitely close planes that pass through the points x and $x + dx$ are parallel to the plane OYZ (Fig. 16.3).

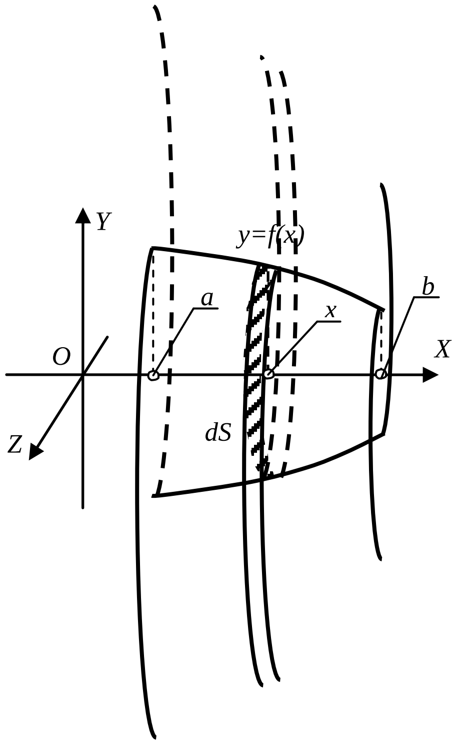

Fig. 16.3

Let's replace the figure formed between the sections of the truncated cone, the generatrix of which is equal to $dl = \sqrt{1 + (y')^2}\, dx$, and the radii of the bases are equal to $y(x)$ and $y(x + dx)$. If the height dx of the cone is small, then the area

dS of the element of the lateral surface of this figure is equal to the area of the lateral surface of the truncated cone: $dS = 2\pi y(x)\sqrt{1+(y')^2}\,dx$. This is the ***differential of the element of the surface area*** of ***rotation***. After its integration in the range from a to b, we find the entire area of the surface of rotation:

$$S = 2\pi\int_a^b y(x)\sqrt{1+(y')^2}\,dx. \qquad (16.4)$$

If the surface is formed by rotation around the axis OX of the curve L defined by parametric equations

$$x = \varphi(t),\ y = \psi(t),\ t \in [\alpha, \beta],$$

then, by replacing the variables under the sign of the definite integral in formula (16.4), we obtain the following expression for the area S of the entire surface:

$$S = 2\pi\int_\alpha^\beta \psi(t)\cdot\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt. \qquad (16.5)$$

Example 16.4. Find the ***surface area*** S of a ***sphere*** with radius R.

The solution. Let the sphere be formed by the rotation of a circle $x^2 + y^2 = R^2$ around an axis OX. Consider part of the circle in the first quarter $y = \sqrt{R^2 - x^2}$, $x \in [0; R]$. By rotating it around the axis OX, we get only half of the sphere, which is located in the half-space R_3 at $x \ge 0$. To calculate the surface area $S_{1/2}$ of this half of the sphere, we apply the formula (16.4), where $y' = -\dfrac{x}{\sqrt{R^2 - x^2}}$; $dl = \dfrac{R\,dx}{\sqrt{R^2 - x^2}}$; $dS = 2\pi R\,dx$.

As a result, we get: $S_{1/2} = 2\pi\int_0^R R\,dx = 2\pi R^2$. To find the surface area of the entire sphere, you need to double the result: $S = 2S_{1/2} = 4\pi R^2$. So, we got the well-known formula for calculating the surface area of the sphere.

Example 16.5. Find the area S of the surface formed by rotation around the axis OX of the **cycloid**, which is determined by the parametric equations:

$$x = a(t - \sin t),\ y = a(1 - \cos t),\ 0 \le t \le 2\pi.$$

The solution. According to formula (16.5), we get:

$$S = 2\pi\int_\alpha^\beta \psi(t)\cdot\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt = 2\sqrt{2}\pi a^2\int_0^{2\pi}(1-\cos t)^{\frac{3}{2}}\,dt = \frac{64}{3}\pi a^2$$

16.2. Application of the definite integral in physical problems

16.2.1. Calculation of the work performed by the variable force when moving a material point

Let a material point M move in a straight line along an axis OX, under the action of some variable force $F = F(x)$, and the vector of action of this force is collinear to the ort $\vec{i}$ axis OX. Then, the work A performed by the force F when moving the point M from position $x = a$ to position $x = b$, is calculated according to the following formula:

$$A = \int_a^b F(x)\,dx. \qquad (16.6)$$

Let`s consider a ***more general case***. Let a material point M under the action of some variable force $\vec{F}(x, y, z)$ move along a spatial curvilinear path L. When the variable force vector $\vec{F}(x, y, z)$ is three-dimensional, e.g.:

$$\vec{F}(x, y, z) = P(x, y, z) \cdot \vec{i} + Q(x, y, z) \cdot \vec{j} + R(x, y, z) \cdot \vec{k},$$

and the elementary displacement vector, which is directed along the tangent to the curve L, is such that $\vec{ds} = dx \cdot \vec{i} + dy \cdot \vec{j} + dz \cdot \vec{k}$ the work of the variable force on the curved path L is determined by the ***curvilinear integral***, which has the following form:

$$A = \int_L (\vec{F} \cdot \vec{ds}) = \int_L P\,dx + Q\,dy + R\,dz. \qquad (16.6^*)$$

We will consider spatial problems of this type in the third semester.

Example 16.6. The movement x of the helical spring under the action of the applied force $\vec{F}$ is proportional to the modulus of the applied force $|\vec{F}| = F$. Calculate the work of force $\vec{F}$ when compressing the spring by 0.05 m, if to compress it by 0.01 m a force of 1 N.

The solution. According to the condition of the problem, the modulus F of force and displacement x of the end of the spring are related by the relation: $F = kx,$ where $k -$ is some proportionality factor. We will express force F in newtons, and displacement x – in meters.

First, we will find the value k. At $x = 0{,}01$ $F = 1$. So, we have the following equation relative to k: $1 = 0{,}01k \Rightarrow k = 100.$

Using formula (16.6), we get:

$$A = \int_0^{0,05} 100x\,dx = 100 \left.\frac{x^2}{2}\right|_0^{0,05} = 0{,}125 \text{ joules}$$

Example 16.7. It is known that the force F with which an electric charge e_1 repels another electric charge e_2 (of the same sign) that is at a distance r from the first is calculated by the following formula:

$$F = k\frac{e_1 e_2}{r^2}.$$

Calculate the work of the force F when moving the charge e_2 from a point M_1 that is far r_1 from the charge e_1 to a point M_2 that is far r_2 from the charge e_1. Assume that the charge e_1 is at the point M_0 chosen as the ***initial reference point***.

The solution. Using formula (16.6), we get:

$$A = \int_{r_1}^{r_2} k\frac{e_1 e_2}{r^2}dr = -k\frac{e_1 e_2}{r}\bigg|_{r_1}^{r_2} = ke_1e_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

If $r_2 \to \infty$ the work A is calculated as follows:

$$A = \lim_{r_2\to\infty} ke_1e_2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{ke_1e_2}{r_1}.$$

If $e_2 = 1$ then $A = k\frac{e_1}{r}$. This last value is called the ***potential*** of the ***force field*** formed by the charge e_1.

Example 16.8. Calculate the work A that must be done to raise a body with a mass m from the surface of the Earth vertically upwards to a height of H, if the radius of the Earth is equal to R. What work A must be done to raise the same body to an infinite height H (i.e $H \to \infty$)?

The solution. According to Newton's law, the module of the force F of attraction of the body by the Earth

$$F(x) = \gamma\frac{mM}{x^2},$$

where M – the mass of the Earth; γ – gravity table; x – the distance from the center of the body to the center of the Earth. Let $\gamma mM = k$, then $F(x) = kx^{-2}$, where $R \le x \le R + H$.

When $x = R$ this force is equal to the weight of the body: $F(R) = mg$, that is

$$\frac{k}{R^2} = mg \Rightarrow k = mgR^2.$$

So, the law of interaction between the body and the Earth has the following form:

$$F(x) = mgR^2x^{-2}.$$

The elementary work that needs to be done to raise a body from the Earth's surface vertically upwards to a height of dx, will be equal to: $dA = F(x)dx$. To find the complete work A, you need to integrate this ***work differential*** in the range from R to $R + H$:

$$A = mgR^2 \int_R^{R+H} x^{-2}dx = \frac{mgRH}{R+H}.$$

If we put $H \to \infty$ in this formula, we get: $A = mgR$.

Example 16.9. Calculate the work that must be done to pump liquid from a conical tank turned upside down. The radius and height of the cone are equal to R and H, respectively, and the density of the liquid is ρ.

The solution. Let's consider the ***elementary liquid layer*** located at a depth of x, a cylinder that has a height dx and a radius y (Fig. 16.4).

Then the ***weight differential*** dP of the elementary liquid layer is equal to $dP = \rho g dV = \rho g \pi y^2 dx$, where ρ – is the density of the liquid, g – is the acceleration of free fall, and dV – is the volume of the elementary cylinder. From the similarity of triangles AOD and CBD, we express y through x: $\frac{R}{y} = \frac{H}{H-x}$. Therefore, the weight differential dP of the elementary liquid layer is equal to: $dP = \frac{\rho g \pi R^2 (H-x)^2 dx}{H^2}$.

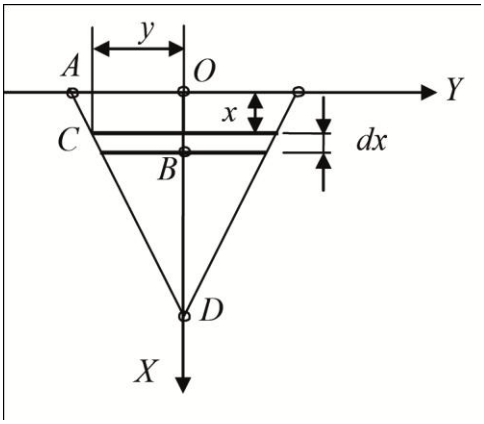

Fig. 16.4

The elementary work dA that must be done to raise this elementary layer of liquid to a height of x, is equal to:

$$dA = \frac{\rho g \pi R^2 (H-x)^2 x dx}{H^2}.$$

This elementary work dA is a ***work differential*** A. Therefore, the total work required to pump out all the liquid from the conical tank is equal to the definite integral calculated over the entire height of the conical tank:

$$A = \int_0^H \frac{\pi\rho g R^2 (H-x)^2 x dx}{H^2} = \frac{\pi\rho g \pi R^2}{H^2} \int_0^H (H-x)^2 x dx = \frac{\pi\rho g R^2 H^2}{12}.$$

16.2.2. Calculation of the pressure force of a liquid on a vertical wall (plate) immersed in it

To calculate the ***force of liquid pressure*** on a horizontal plate, we will use the well-known law of B. Pascal. According to it, the force of liquid pressure P on a horizontal plate is equal to its area S multiplied by the depth h of immersion, the density ρ of the liquid and the acceleration g of free fall: $P = \rho g h S$ [8; 15; 19].

If the plate is immersed in the liquid at a certain angle, then its different points will lie at different depths, and this formula cannot be used. However, if the plate is very small vertically, then all its points lie ***at almost the same depth***, which is considered the depth of immersion of the plate. This makes it possible to divide the plate vertically into horizontal elementary areas (***layers***), find the ***pressure force differential*** on the elementary area of this plate, and then calculate the pressure force on its entire surface by ***integration along the vertical coordinate***.

Example 16.10. Find the pressure force P on a vertical plate that has the shape of an equilateral trapezoid $ABCD$ (Fig. 16.5) with an upper base a, a lower base b and a height h and which is immersed in a liquid to a depth c. The density of the liquid is ρ.

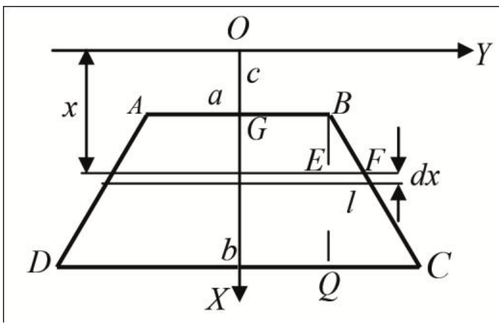

Fig. 16.5

The solution. Let the horizontal elementary platform be at depth x (Fig. 16.5). Let's introduce the following notations: $AB = a$, $DC = b$, $QB = h$, $OG = c$, $EF = l(x)$.

The ***area of the horizontal elementary platform*** is calculated as follows:

$$dS = (a + 2l)dx,$$

where $l(x) = \frac{b-a}{2h}(x-c)$ – is determined from the similarity of triangles BQC and BEF.

Then the expression for the area of the horizontal elementary platform takes the following form:

$$dS = \left(a + \frac{b-a}{h}(x-c) \right) dx.$$

Now, on the basis of Pascal's law, we will write down the expression for the pressure force on this horizontal elementary platform (***pressure force differential***):

$$dP = \rho g \left(a + \frac{b-a}{h}(x-c) \right) x dx.$$

After that, we can find the force of fluid pressure on the entire vertical plate $ABCD$ if we integrate the last expression for dP by the variable x in the range from c to $c+h$:

$$P = \rho g \int_{c}^{c+h} \left(a + \frac{b-a}{h}(x-c) \right) x dx = \left[\frac{a+b}{2} ch + \frac{h^2}{6}(a+2b) \right] \rho g.$$

If the parameters of the problem are chosen as follows: $a = c = h = 1$ m, $b = 2$ m, then the magnitude of the pressure force P on the plate will be equal to $P = \frac{7}{3}\rho g$ (newtons).

16.2.3. Calculation of the center of mass of a non-homogeneous rod

Let's recall the problem of calculating the mass of a non-homogeneous rod, presented in the lecture material. There was derived a formula for calculating the mass of the rod through the limit of integral sums in the following form ($\rho(x)$ – linear density of the rod material):

$$m = \lim_{\lambda \to 0} \sum_{i=1}^{n} \rho(\xi_i) \cdot \Delta x_i \text{, де } \lambda = \max_{i=\overline{1,n}} \Delta x_i . \tag{16.7}$$

If we go to the limit in formula (16.7), then the mass of the non-homogeneous rod is calculated through the definite integral:

$$m = \int_a^b \rho(x)dx . \tag{16.8}$$

To find the center of mass of the rod, we will again use the method of integral sums and the well-known formula for calculating the coordinates x_c of the center of mass of the system $\{m_i(x_i)\}$ of material points that have masses m_i and are located at the points x_i of the axis OX . Let's write this formula:

$$x_c = \frac{m_1x_1 + m_2x_2 + \cdots + m_nx_n}{m_1 + m_2 + \cdots + m_n} = \frac{\sum_{i=1}^{n} m_ix_i}{\sum_{i=1}^{n} m_i} . \tag{16.9}$$

Let's consider an arbitrary division of the segment $[a,b]$ by points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b .$$

Let's calculate the mass m_i of the part of the rod located on the elementary segment $[x_{i-1}, x_i]$. Using formula (16.8) for the elementary segment $[x_{i-1}, x_i]$, we obtain: $m_i = \int_{x_{i-1}}^{x_i} \rho(x)dx$. Let's apply the Theorem about the ***average value of a function*** on a segment $[x_{i-1}, x_i]$: $m_i = \rho(\xi_i)\Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$.

Assuming that the mass m_i of an elementary segment $[x_{i-1}, x_i]$ is completely concentrated in some point $\xi_i \in [x_{i-1}, x_i]$, one can identify a non-homogeneous rod with a system of material points with masses m_i

$$\sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \rho(x)dx = \int_a^b \rho(x)dx = m,$$

then, using formula (16.9), we find an approximate expression for the coordinate x_c of the center of mass of a non-homogeneous rod

$$x_c \approx \frac{\sum_{i=1}^{n} \xi_i \rho(\xi_i)\Delta x_i}{m} . \tag{16.10}$$

The expression in the numerator of the right-hand side of relation (16.10) represents the ***integral sum*** for the function $x\rho(x)$ on the segment $[a,b]$.

Therefore, in accordance with the above considerations and moving to the limit of the integral sum in the numerator (16.10), we determine the coordinate x_c of the center of mass of the non-homogeneous rod by the formula

$$x_c = \int_a^b x\rho(x)dx \Big/ \int_a^b \rho(x)dx \,. \qquad (16.11)$$

Example 16.11. Find the mass of a rod 100 cm long, if its linear density varies according to the law of $\rho(x) = (12x + 0,24x^2)$ g/cm, where $x -$ is the distance from one of the ends of the rod, which is located at the origin. Find the coordinate of the center of mass of the rod.

The solution. Calculate the mass of the rod using formula (16.8):

$$m = \int_0^{100} \rho(x)dx = \int_0^{100} (12x + 0,24x^2)dx = \left(6x^2 + 0,08x^3\right)\Big|_0^{100} = 140 \text{ kg.}$$

Now we will find the value of the numerator written in formula (16.11):

$$J = \int_0^{100} x\rho(x)dx = \int_0^{100} x(12x + 0,24x^2)dx = \left(4x^3 + 0,06x^4\right)\Big|_0^{100} = 10^7 \text{ g* cm.}$$

According to formula (16.11), we calculate the coordinate x_c of the center of mass of the rod:

$$x_c = \frac{\int_a^b x\rho(x)dx}{\int_a^b \rho(x)dx} = \frac{10^7}{14 \cdot 10^4} = 71,43 \text{ cm.}$$

16.2.4. Calculation of CM of symmetric flat figures

Example 16.12. A homogeneous plate with a constant surface density $\rho(x, y) = \rho = const$ lying in a plane OXY, bounded by a curve $y = 4 - x^2$ and an axis OX. Find the coordinates of the CM of this plate (Fig. 16.6).

The solution. We denote the area formed by this plate in the plane OXY by D. We work according to well-known formulas from physics (mechanics), which establish the dependence of the location of the CM of a flat body in the plane OXY depending on the geometric and mass characteristics of the body:

$$y_C = \frac{\int_c^d y \cdot dm}{m};\; x_C = \frac{\int_a^b x \cdot dm}{m};\; m = \int_a^b dm = \rho\int_a^b y(x)dx = \rho S_D \,. \qquad (16.12)$$

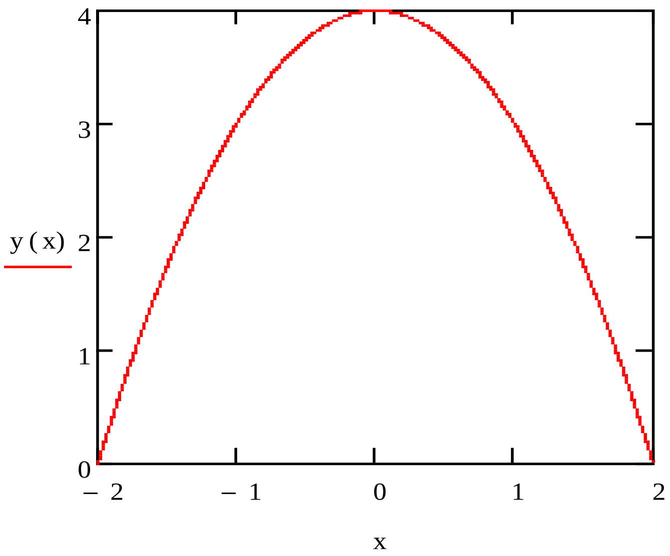

Fig. 16.6

The integrals in the numerators of formulas (16.12) are called ***static moments*** relative to the corresponding coordinate axes: $M_{OX} = \int_c^d y \cdot dm$ - static moment of the plate relative to the axis OX, and $M_{OY} = \int_a^b x \cdot dm$ – static moment of the plate relative to the axis OY. Since the plate is symmetrical about the axis OY and is homogeneous, we have: $x_C = 0$. So you only need to find y_C.

Let's write down the expression for the mass element dm. We will divide our plate vertically into horizontal strips, and for a certain value y we will choose one of these strips with height dy. We consider that each elementary strip is a rectangle. The width of this rectangle at the height y will be equal to: $x = 2\sqrt{4-y}$. Obviously, this width is variable if the ordinate varies from 0 to 4. Therefore, to find the formula for the elemental mass dm of this strip, you need to multiply the surface density $\rho = const$ by the area element dS, namely:

$$dm = \rho dS = 2\rho\sqrt{4-y}\,dy. \qquad (16.13)$$

Substitute expression (16.13) into formula (16.12) for $M_{OX} = \int_c^d y \cdot dm$, where the integration interval is as follows $[0,4]$:

$$M_{OX} = 2\rho\int_0^4 y \cdot \sqrt{4-y}\,dy = \frac{256\rho}{15}.$$

Now it remains to find the mass of the plate:

$$m = \int_{-2}^{2} dm = 2\rho\int_0^2 (4-x^2)dx = \frac{32\rho}{3}.$$

For the calculation y_C, it remains M_{ox} to divide by the mass m:

$$y_C = \frac{256}{15} \cdot \frac{3}{32} = \frac{8}{5} \text{ (linear units).}$$

Example 16.13. Find the mass of the body bounded by the single-cavity hyperboloid $\frac{x^2}{3^2} + \frac{y^2}{4^2} - \frac{z^2}{2^2} = 1$, the planes $z = 0$ and $z = 1$. Assume that the density of the material of the body is given by the formula $\rho = z^2$.

The solution. Make a drawing of the body yourself. We will use the same approach as at the beginning of this practical session. Let's cut the body into planes parallel to the plane $z = 0$, and calculate the mass of the elementary layer of such a division. Let the thickness of such a layer be equal to dz. The boundary of each such section of the body will represent an ellipse, the equation of which has the form

$$\frac{x^2}{3^2\sqrt{\frac{z^2}{2^2}+1}} + \frac{y^2}{4^2\sqrt{\frac{z^2}{2^2}+1}} = 1,$$

the variable semi-axis of which are aligned accordingly: $a = 3 \cdot \sqrt{\frac{z^2}{2^2}+1}$ and $b = 4 \cdot \sqrt{\frac{z^2}{2^2}+1}$. It is known that the area of a flat region bounded by an ellipse is equal to:

$$S(z) = \pi ab = \pi \cdot 3 \cdot \sqrt{\frac{z^2}{2^2}+1} \cdot 4 \cdot \sqrt{\frac{z^2}{2^2}+1} = 6\pi(z^2 + 4).$$

Considering the elementary layer of the body as a straight elliptical cylinder with height dz, we calculate its elementary volume:

$$dV = S(z)dz = 6\pi(z^2 + 4)dz.$$

Now you can find an expression for the mass element

$$dm = \rho(z)dV = 6\pi z^2(z^2 + 4)dz$$

and calculate the mass m of the entire body by integrating it along the vertical coordinate z as follows:

$$m = \int_0^1 S(z)\rho(z)dz = \int_0^1 6\pi z^2(z^2 + 4)dz = \frac{46\pi}{5}.$$

Tasks for classroom and independent work

16.1. The ellipse $4x^2 + 9y^2 = 36$ rotates around the axis a) OX; b) OY. Find the volumes of the corresponding ellipsoids of rotation that are formed.

16.2. Find the volume of the body formed by rotation around the axis OX of the figure bounded by the lines $2y = x^2$ and $2x + 2y - 3 = 0$.

16.3. Find the volume of the body formed by the rotation of the figure bounded by the lines $y = 2x - x^2$ and $y = 0$ around the axis: a) OX; b) OY.

16.4. A semi-infinite curvilinear trapezoid, which is formed by a curve $y = \sqrt{x}e^{-x}$ and its asymptote, rotates around the axis OX. Find the volume of the body formed by such rotation.

16.5. The curve $y = e^{-x}$ $(x \geq 0)$ rotates around the axis OY. Find the volume of the body formed by such a rotation.

16.6. An infinite curvilinear trapezoid, which is formed by the curve $x = \dfrac{1}{y^2+1}$ and its asymptote, rotates around the axis OY. Find the volume of the infinite "spindle" formed during such rotation.

16.7. Find the volume of the body formed by rotation around the axis OX of the figure bounded by one arch of the cycloid $x = a(t - \sin t),\ y = a(1 - \cos t);\ 0 \leq t \leq 2\pi$ and $y = 0$.

16.8. Find the volume of the body formed by the rotation of the loop area of the curve $x = 2t - t^2,\ y = 4t - t^3$ around the axis: a) OX; b) OY.

16.9. Find the volume of the body formed by rotation around the polar axis of the figure bounded by the closed curve $\rho = a\sin^3\dfrac{\varphi}{2}$.

16.10. Find the area of the surface of rotation around the axis OX of the curved trapezoid bounded by the curve $y = \sin x$ and the axis OX on the segment $x \in [0, \pi]$.

16.11. Find the surface area of the part of the paraboloid formed by the rotation around the OX axis of the parabola $y^2 = 4ax,\ 0 \leq x \leq 3a$.

16.12. Find the area of the surface formed by the rotation of one arch of the cycloid $x = a(t - \sin t),\ y = a(1 - \cos t);\ 0 \leq t \leq 2\pi$ around the axis OY.

16.13. What work must be done to stretch the spring by 4 cm, if it is stretched by 1 cm by a force of 1 N?

16.14. Calculate the work that must be done when oil is pumped through the upper hole from the tank. The tank is completely filled, has the shape of a cylinder with a horizontal axis, the density of oil ρ, the length of the tank h and the radius R.

16.15. Calculate the work that must be done to pump liquid bitumen through the top of a hemispherical fairy of radius R (the density of liquid bitumen ρ).

16.16. Find the force of pressure on a vertical plate immersed in water in the form of an equilateral triangle with height h. The base of the triangle is equal a and lies on the surface of the water.

16.17. Solve the same problem 16.16, only under the condition that the top of the triangle is located on the surface of the water, and the base of the triangle is parallel to the surface of the water.

16.18. Find the mass and coordinate of the center of mass of a rod 100 cm long, if the linear density of the material of the rod varies along its length according to the law of $\rho(x) = (20x + 0{,}15x^2)$ g/cm, where x – is the distance from one of the ends of the rod, which is located at the origin of the coordinates.

16.19. The speed of movement of a body thrown vertically upward with an initial speed v_0, without taking into account air resistance, is equal to $v = v_0 - gt$, where t – time, g – is the acceleration of free fall. To what maximum height will the body rise?

16.20. The speed of movement of the body has the form: $v(t) = 0{,}1e^{-0{,}02t}$ (m/s). Find the path that the body will take from the beginning of the movement to its complete stop.

16.21. A homogeneous plate with a surface density $\rho(x, y) = \rho = const$ lying in a plane OXY, bounded by a curve $y = \sqrt{R^2 - x^2}$ and an axis OX. Find the coordinates of the CM of this plate.

16.22. Find the mass of the body bounded by an unlimited elliptic paraboloid $\frac{x^2}{4^2} + \frac{y^2}{5^2} - z = -1$. Assume that the density of the material of the body is given by the formula $\rho = \frac{1}{z^3 + 1}$.

16.23. Find the mass m of a body bounded by a two-cavity hyperboloid $\frac{x^2}{5^2} + \frac{y^2}{2^2} - z^2 = -1$, provided that $z \geq 1$. Assume that the density of the material of the body is given by the formula $\rho = ze^{-z^2}$.

16.24. Find the volume of the body of revolution of the plane region $D: \{0 \leq \varphi \leq 2\pi; \rho(\varphi) = a(1 + \cos\varphi)\}$ around the polar axis $\varphi = 0$.

16.25. Prove that in the ***polar coordinate system*** the volume of a body of rotation around the polar axis of a curved sector

$D:\{0 \le \alpha \le \varphi \le \beta \le \pi; 0 \le \rho \le \rho(\varphi)\}$, where (φ, ρ) – are the polar coordinates, calculated by the formula:

$$V = \frac{2\pi}{3}\int_{\alpha}^{\beta} \rho^3(\varphi)\sin\varphi\, d\varphi .$$

Answers and instructions

16.1. a) $\frac{4}{3}\pi ab^2$; b) $\frac{4}{3}\pi a^2 b$. **16.2.** $\frac{272}{15}\pi$. **16.3.** a) $\frac{16}{15}\pi$; b) $\frac{8}{3}\pi$.

16.4. $\frac{\pi}{4}$. **16.5.** 2π. **16.6.** $\pi^2/2$. **16.7.** $5\pi^2 a^3$.

16.8. a) $\frac{64}{35}\pi$; b) $\frac{64}{105}\pi$. **16.9.** $\frac{8}{33}\pi a^3$. **16.10.** $2\pi(\sqrt{2} + \ln(1+\sqrt{2}))$.

16.11. $\frac{56}{3}\pi a^2$. **16.12.** $16\pi^2 a^2$. **16.13.** $\frac{1}{8}$. **16.14.** $\pi\rho ghR^3$.

16.15. $\frac{\pi\rho gR^4}{4}$. **16.16.** $\frac{ah^2}{6}$. **16.17.** $\frac{ah^2}{3}$. **16.18.** 69,44 cm.

16.19. $\frac{v_0^2}{2g}$. **16.20.** 250 m. **16.21.** $\left(0; \frac{4R}{3\pi}\right)$. **16.22.** $\frac{20\pi}{3}\ln(4)$.

16.23. $m = \frac{5\pi}{e}$. **16.24.** $V = 8/3\,\pi a^3$.

Topic III. Improter integrals of the first and second kind

PRACTICAL LESSON 17
Improper integrals of the first kind

17.1. Basic concepts and definitions, geometric meaning. Cauchy criterion

Improper integrals of the first kind are also called integrals with ***infinite limits of integration***.

Let the function $f(x)$ be defined and ***continuous*** for any values of $x \in [a;+\infty)$. Let us consider the definite integral $F(b) = \int_a^b f(x)dx$ with a variable upper limit b. This integral has meaning for any value of $b > a$. When the upper limit b changes, the integral $F(b)$ changes and is a ***continuous function*** of its variable upper limit b. Consider the behavior of this integral at $b \to +\infty$ (Fig. 17.1).

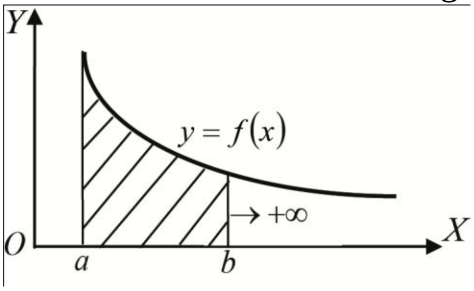

Fig. 17.1

Definition. *The* ***improper integral*** *of the first kind of a function* $f(x)$ *on the interval* $a \le x < +\infty$ *is called the limit of the integral* $\int_a^b f(x)dx$ *at* $b \to +\infty$ *and is denoted by the symbol*

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx . \qquad (17.1)$$

If the limit (17.1) exists, then it is said that the improper integral $\int_a^{+\infty} f(x)dx$ exists or is ***convergent***. If the integral $\int_a^b f(x)dx$ at $b \to +\infty$ ***has no finite limit***, it is said that the improper integral $\int_a^{+\infty} f(x)dx$ ***does not exist or is divergent***.

Let's find out the ***geometric meaning of the improper integ*****ral** of the first kind. Consider the case when $f(x) \ge 0$. If the integral $\int_a^b f(x)dx$ is equal to the area of the region bounded by the curve $y = f(x)$, the abscissa axis and the ordinates $x = a$, $x = b$, then it is natural to assume that the improper integral $\int_a^{+\infty} f(x)dx$ is equal to the area of the unbounded (infinite) region contained between the lines: $y = f(x)$, $x = a$, $x \to \infty$ and $y = 0$ (Fig. 17.1).

Analogously to the integral $\int_a^{+\infty} f(x)dx$, the concept of an improper integral of the first kind with an infinite ***lower limit*** (that is, on the interval $(-\infty, b]$) is introduced:

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx . \qquad (17.2)$$

The improper integral with ***two infinite limits*** is defined by the equality:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx , \qquad (17.3)$$

where c – is an arbitrary real number.

The integral on the left in formula (17.3) ***exists or is convergent*** only when both integrals on the right are convergent.

Remark. It follows from the definition that ***the improper integral is the limit of a definite integral*** with a ***variable limit of integration***.

The Cauchy Criterion is used to study improper integrals of the first kind.

Theorem 17.1 (*Cauchy Criterion*). *In order for the improper integral (17.1) to be **convergent**, it is **necessary and sufficient** for an arbitrary number $\varepsilon > 0$ to find a number $\Delta \ge a$ such that for any numbers $b_1 > \Delta$ and $b_2 > \Delta$ the inequality holds*

$$\left| \int_{b_1}^{b_2} f(x)dx \right| < \varepsilon . \qquad (17.4)$$

Formula (17.4) establishes that in order for the improper integral (17.1) to converge, it is ***necessary and sufficient*** that its "***tail***" (17.4) be ***infinitely small*** (that is, smaller than an arbitrary positive number) if both limits of integration are ***moved to infinity***.

Let us consider the simplest examples of integration of improper integrals of the 1st kind.

Examples 17.1.

a) $\int\limits_0^{+\infty} \frac{dx}{1+x^2}$;

b) $\int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^2}$;

c) $\int\limits_{-\infty}^{0} \cos 2x dx$;

d) $\int\limits_0^{+\infty} e^x dx$;

e) $\int\limits_{2/\pi}^{\infty} \frac{1}{x^2} \cdot \sin\left(\frac{1}{x}\right) dx$;

f) $\int\limits_1^{\infty} \frac{\ln x}{x^2} dx$.

The solution.

a) By definition, we have:

$$\int\limits_0^{+\infty} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \int\limits_0^{b} \frac{dx}{1+x^2} = \lim_{b\to+\infty} \operatorname{arctg} x\Big|_0^b = \lim_{b\to+\infty} \operatorname{arctg} b - \operatorname{arctg} 0 = \frac{\pi}{2}.$$

b) Using the parity of the integral function and the result of the previous task, we get

$$\int\limits_{-\infty}^{+\infty} \frac{dx}{1+x^2} = 2\int\limits_0^{+\infty} \frac{dx}{1+x^2} = 2\cdot\frac{\pi}{2} = \pi.$$

The given integral is ***convergent***.

c) By definition, we have:

$$\int\limits_{-\infty}^{0} \cos 2x dx = \lim_{a\to-\infty} \int\limits_a^{0} \cos 2x dx = \lim_{a\to-\infty} \frac{1}{2}\sin 2x\Big|_a^0 = -\lim_{a\to-\infty} \frac{1}{2}\sin 2a.$$

The limit $\sin 2a$ at $a \to -\infty$ ***does not exist***, so this integral is ***divergent***.

d) Let's calculate according to the definition:

$$\int\limits_0^{+\infty} e^x dx = \lim_{b\to\infty} e^x\Big|_0^b = \lim_{b\to+\infty} e^b - 1 = +\infty.$$

e) Let us show that the given integral is ***convergent***.

$$\int\limits_{2/\pi}^{\infty} \frac{1}{x^2}\cdot\sin\left(\frac{1}{x}\right)dx = -\int\limits_{2/\pi}^{\infty} \sin\left(\frac{1}{x}\right)d\left(\frac{1}{x}\right) = \lim_{b\to+\infty}\left(\cos\left(\frac{1}{x}\right)\right)\Bigg|_{2/\pi}^{b} = \lim_{b\to+\infty}\left(\cos\left(\frac{1}{b}\right)\right) - \cos\frac{\pi}{2} = 1 - 0 = 1.$$

f) By definition, we have:

$$\int_1^{\infty}\frac{\ln x}{x^2}dx=-\int_1^{\infty}\ln x\cdot d\left(\frac{1}{x}\right)=-\left(\frac{\ln x}{x}\right)\Bigg|_1^{\infty}+\int_1^{\infty}\frac{dx}{x^2}=-\left(\frac{1}{x}\right)\Bigg|_1^{\infty}=1.$$

The given integral is ***convergent***.

Example 17.2. Let’s clarify the question at which values of the exponent $\lambda>0$ the improper integral $\int_a^{+\infty}\frac{dx}{x^{\lambda}}$, $(a>0)$, is ***convergent***. It is often used for comparison with other improper integrals ***as a test***.

The solution. Let $\lambda\neq 1$, then

$$\int_a^b x^{-\lambda}dx=\frac{1}{1-\lambda}x^{1-\lambda}\Big|_a^b=\frac{1}{1-\lambda}\left(b^{1-\lambda}-a^{1-\lambda}\right).$$

$$\lim_{b\to+\infty}\frac{1}{1-\lambda}\left(b^{1-\lambda}-a^{1-\lambda}\right)=\begin{cases}\infty, if & \lambda<1;\\ \dfrac{a^{1-\lambda}}{\lambda-1}, if & \lambda>1\end{cases}.$$

Let $\lambda=1$, then

$$\lim_{b\to+\infty}\int_a^b\frac{dx}{x}=\lim_{b\to+\infty}\ln x\Big|_a^b=\lim_{b\to+\infty}\left(\ln b-\ln a\right)=\infty.$$

Therefore, when $\lambda\le 1$ the given integral is ***divergent***, when $\lambda>1$ – is ***convergent***.

17.2. Comparison signs for improper integrals of the first kind

Theorem 17.2 (*sign of comparison by inequality*). *If for two functions* $f(x)$ *and* $g(x)$ *for all* x, $(x\ge a)$ *the inequality holds*

$$0\le f(x)\le g(x), \tag{17.5}$$

then the convergence of the integral $\int_a^{+\infty}f(x)dx$ *follows from the convergence of the integral* $\int_a^{+\infty}g(x)dx$*; and conversely, the divergence of the integral* $\int_a^{+\infty}g(x)dx$ *follows from the divergence of the integral* $\int_a^{+\infty}f(x)dx$.

Theorem 17.2 has the following geometric meaning (Fig. 17.2): if the area of the larger unlimited region is a ***finite*** number, then the area of the smaller region is also ***a finite*** number; if the area of a smaller region is an infinitely large value, then the area of a larger region is also an ***infinitely*** large value.

Example 17.3. Check whether the integral is convergent $\int_1^{\infty} \frac{dx}{x^2(e^x+1)}$.

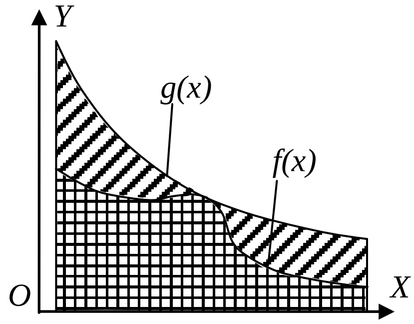

Fig. 17.2

The solution. Let’s choose a test integral $\int_1^{+\infty} \frac{dx}{x^2}$ for comparison. Note that when $x \ge 1$, the inequality $\frac{1}{x^2(e^x+1)} < \frac{1}{x^2}$ holds. The ***test integral*** $\int_1^{+\infty} \frac{dx}{x^2}$ is convergent because

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{A \to +\infty} \int_1^{A} \frac{dx}{x^2} = \lim_{A \to +\infty} \left(-\frac{1}{x} \right) \Bigg|_1^A = \lim_{A \to +\infty} \left(-\frac{1}{A} + 1 \right) = 1 .$$

Therefore, according to ***Theorem 17.2***, the given integral is also convergent, and its value is less than 1.

Example 17.4. Investigate the convergence of the integral $\int_1^{+\infty} \frac{x+\sin^2 x}{\sqrt{x^3}} dx$.

The solution. Note that $\frac{x+\sin^2 x}{\sqrt{x^3}} > \frac{x}{\sqrt{x^3}} = \frac{1}{\sqrt{x}}$.

However, the test integral $\int_1^{+\infty} \frac{dx}{\sqrt{x}}$ is ***divergent***:

$$\int_1^{+\infty} \frac{dx}{\sqrt{x}} = \lim_{A \to +\infty} 2\sqrt{x} \Big|_1^A = +\infty .$$

Therefore, the given integral is also ***divergent***.

Example 17.5*. Investigate the convergence of the ***Fresnel integral*** $\int_0^{+\infty} \sin(x^2) dx$.

The solution. Let's replace the variables $t = x^2,\ x = \sqrt{t},\ dx = \dfrac{dt}{2\sqrt{t}}$. At the same time, the limits of integration will not change. Then the integral will take the following form:

$$\int_0^{+\infty} \sin(x^2)dx = \frac{1}{2}\int_0^{+\infty} \frac{\sin t}{\sqrt{t}}dt .$$

Using additivity, we break the integral into the sum of two integrals:

$$\int_0^{+\infty} \frac{\sin t}{\sqrt{t}}dt = \int_0^{\pi/2} \frac{\sin t}{\sqrt{t}}dt + \int_{\pi/2}^{+\infty} \frac{\sin t}{\sqrt{t}}dt . \qquad (17.6)$$

The first of the two integrals on the right-hand side of expression (17.6) is an ordinary definite integral, since $\dfrac{\sin t}{\sqrt{t}} \to 0$ by $t \to 0$. We apply integration by parts to the second integral from (17.6):

$$\int_{\pi/2}^{+\infty} \frac{\sin t}{\sqrt{t}}dt = \left| \begin{matrix} u = \dfrac{1}{\sqrt{t}};\ dv = \sin t dt; \\ du = -\dfrac{dt}{2t^{3/2}};\ v = -\cos t \end{matrix} \right| = -\frac{\cos t}{\sqrt{t}}\Bigg|_{\pi/2}^{+\infty} - \frac{1}{2}\int_{\pi/2}^{+\infty} \frac{\cos t dt}{t^{3/2}} = -\frac{1}{2}\int_{\pi/2}^{+\infty} \frac{\cos t dt}{t^{3/2}} . \qquad (17.7)$$

The last integral in expression (8) is convergent, since the following inequality holds

$$\frac{\cos t}{t^{3/2}} \le \frac{1}{t^{3/2}} ,$$

and the integral $\displaystyle\int_{\pi/2}^{+\infty} \frac{dt}{t^{3/2}}$ is ***convergent***. Therefore, the integral $\displaystyle\int_{\pi/2}^{+\infty} \frac{\cos t dt}{t^{3/2}}$ is also ***convergent*** by the sign of comparison by inequality, and accordingly, the given ***Fresnel integral*** is also ***convergent***.

Theorem 17.3 (*limit sign of comparison*). *If there is a limit*

$$\lim_{x \to \infty} \frac{g(x)}{f(x)} = K\ ,\ 0 < K < +\infty\ ,\ (\ f(x) > 0,\ g(x) > 0\ ), \qquad (17.8)$$

then the convergence (divergence) of one of the integrals $\displaystyle\int_a^{+\infty} g(x)dx$, $\displaystyle\int_a^{+\infty} f(x)dx$ *implies the convergence (divergence) of the second. Thus, when* $0 < K < +\infty$ *both integrals are* ***simultaneously convergent****, or both are* ***simultaneously divergent****.*

Example 17.6. Investigate the given integral for convergence

$$\int_1^{+\infty} \ln\left(\frac{x^2+2}{x^2+1}\right)dx .$$

The solution. Let's use Theorem 17.3 and consider the limit of the relationship between the given function $\ln\left(\frac{x^2+2}{x^2+1}\right)$ and the test function $\frac{1}{x^2}$. As a result, we have:

$$\lim_{x\to\infty}\left[x^2\ln\left(\frac{x^2+2}{x^2+1}\right)\right]=\lim_{x\to\infty}\left[x^2\ln\left(1+\frac{1}{x^2+1}\right)\right]=\left|\begin{matrix}\ln(1+\alpha)\sim\alpha, \\ if \quad \alpha\to 0\end{matrix}\right|=\lim_{x\to\infty}\left[\frac{x^2}{x^2+1}\right]=1 .$$

Since the test integral $\int_1^{\infty}\frac{1}{x^2}dx$ is ***convergent***, the given integral is ***also convergent*** by the limit sign of comparison.

Example 17.7. Investigate the given integral for convergence $\int_1^{+\infty}\left(1-\cos\frac{2}{x}\right)dx$.

The solution. Let's use Theorem 17.3 and consider the limit of the relationship between the given function $1-\cos\frac{2}{x}$ and the test function $\frac{1}{x^2}$. As a result, we have:

$$\lim_{x\to+\infty}\frac{1-\cos\frac{2}{x}}{\frac{1}{x^2}}=\left|\begin{matrix}\cos\frac{2}{x}\sim 1-\frac{2}{x^2}+\cdots . \\ if \;\; x\to\infty\end{matrix}\right|=\lim_{x\to+\infty}\frac{\frac{2}{x^2}}{\frac{1}{x^2}}=2 .$$

Since the test integral $\int_1^{\infty}\frac{1}{x^2}dx$ is ***convergent***, the given integral is ***also convergent*** by the limit sign of comparison.

17.3. The concept of absolute and conditional convergence improper integrals of the first kind

In ***Theorems 17.2 and 17.3***, integrals of integral functions were considered. For a function $f(x)$ that changes sign on an infinite interval, the following Theorem holds:

Theorem 17.4. *If the integral* $\int_a^{+\infty} |f(x)|dx$ *– is convergent, then the integral* $\int_a^{+\infty} f(x)dx$ *is also convergent.*

The proof of Theorem 17.4 follows immediately from the inequality: $f(x) \le |f(x)|$. Thus, by the sign of ***comparison by inequality*** (Theorem 17.2), the convergence of the improper integral of the function $|f(x)|$ follows from the convergence of the improper integral of the function $f(x)$. So, we have:

$$\int_a^{+\infty} f(x)dx \le \int_a^{+\infty} |f(x)|dx .$$

Definition. *If, together with a convergent integral* $\int_a^{+\infty} f(x)dx$, *the integral* $\int_a^{+\infty} |f(x)|dx$ *is also convergent, then the integral* $\int_a^{+\infty} f(x)dx$ *is called* ***absolutely convergent***.

Definition. *If the integral* $\int_a^{+\infty} f(x)dx$ *is convergent and the integral* $\int_a^{+\infty} |f(x)|dx$ *is divergent, then the integral* $\int_a^{+\infty} f(x)dx$ *is called* ***conditionally convergent***.

Example 17.8. Investigate the convergence of the integral $\int_1^{+\infty} \frac{\sin x dx}{x^3}$.

The solution. Here the integrand function changes sign. Note that $\left|\frac{\sin x}{x^3}\right| < \left|\frac{1}{x^3}\right|$, but the test integral $\int_1^{+\infty} \frac{dx}{x^3}$ is convergent, since:

$$\int_1^{+\infty}\frac{dx}{x^3}=\lim_{A\to+\infty}\int_1^{A}\frac{dx}{x^3}=\lim_{A\to+\infty}\left(-\frac{1}{2}x^{-2}\right)\Bigg|_1^A=-\frac{1}{2}\lim_{A\to+\infty}\left(A^{-2}\right)+\frac{1}{2}=\frac{1}{2}.$$

Therefore, the test integral is ***convergent***, and therefore the given integral is also ***convergent***, and ***absolutely***.

In some cases, it is necessary to investigate the convergence of improper integrals in which the integrand is the product of two functions. In this case, the following theorem will be useful.

Theorem 17.5 (*Dirichlet's sign for studying the convergence of the integral* $\int_a^{\infty} f(x)g(x)dx$).

Let the following conditions be fulfilled:

1) *The function* $f(x)$ *is* ***integral*** *in any interval* $[a,b]\subset[a,+\infty)$, *and the integral*

$$\left|\int_a^b f(x)dx\right|\le M$$

is ***bounded*** *for any* $b>a$.

2) *The function* $g(x)$ *is* ***continuously differentiable*** *with respect to* $x\ge a$ *and,* ***monotonically decreasing****, goes to zero with respect to* $x\to+\infty$ $(\lim_{x\to+\infty} g(x)=0)$.

Then the improper integral

$$\int_a^{+\infty} f(x)g(x)dx \qquad (17.9)$$

is ***convergent***.

Example 17.9. Investigate the important Dirichlet integral for convergence:

$$F(a)=\int_a^{\infty}\frac{\sin x}{x}dx.$$

The solution. The given integral has the form (17.9). Let's put $f(x)=\sin x$, and $g(x)=x^{-1}$. It is obvious that $\left|\int_a^A \sin x dx\right|=\left|\cos a-\cos A\right|\le 2$, and $\lim_{x\to\infty} g(x)=\lim_{x\to\infty}\frac{1}{x}=0$, at the same time $g(x)=x^{-1}$ ***monotonically goes to zero***. Therefore, the given integral is ***convergent*** and if $a=0$, then it is equal to $F(0)=\pi/2$. This result was ***first obtained*** by ***Dirichlet***.

17.4. The concept of the principal value of improper integrals of the first kind

Let the function $f(x)$ be defined on the entire number line $(-\infty, +\infty)$ and integrable on any segment $[a,b] \subset (-\infty, +\infty)$.

Definition. *If there is a limit*

$$\lim_{R \to \infty} \int_{-R}^{R} f(x)dx ,$$

then it is called the ***principal value*** *of the* ***improper integral*** *and is denoted as follows*:

$$V.p. \int_{-\infty}^{+\infty} f(x)dx .$$

Certificate. Notation $V.p.$ means an abbreviated record of a phrase «***Valeur principal***».

So, by the definition of the ***principal*** value, we have:

$$V.p. \int_{-\infty}^{+\infty} f(x)dx = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx . \qquad (17.10)$$

Example 17.10. Find the ***principal value*** of the integral $\int_{-\infty}^{+\infty} \frac{1+x}{1+x^2} dx$.

The solution. It is not difficult to check that the improper integral is ***divergent in the usual sense***. But,

$$V.p. \int_{-\infty}^{+\infty} \frac{1+x}{1+x^2} dx = \lim_{R \to +\infty} \left(\operatorname{arctg} x + \frac{1}{2} \ln(1+x^2) \right) \Bigg|_{-R}^{R} = \lim_{R \to +\infty} 2 \operatorname{arctg} R = 2 \cdot \frac{\pi}{2} = \pi .$$

Tasks for classroom and independent work

I. Calculate improper integrals:

17.1. $\int_{0}^{+\infty} \frac{arctgx}{(1+x^2)^{3/2}} dx$.

17.2. $\int_{3}^{\infty} \frac{dx}{x^3 \sqrt{x^2 - 9}}$.

17.3. $\int_{1}^{\infty} \frac{\operatorname{arctg} x}{x^7} dx$.

17.4. $\int_{1}^{\infty} \frac{dx}{x(x-1)^{2/3}}$.

17.5. $\int_0^{+\infty} \ln\left(\frac{x^2+2}{x^2+1}\right)dx$.

17.6. $\int_0^{\infty} \frac{x^2+1}{x^4+1}dx$.

17.7. $\int_1^{\infty} \frac{dx}{x\sqrt{x^{10}+x^5+1}}$.

17.8. $\int_0^{\infty} \frac{x\ln x}{(x^2+1)^2}dx$.

17.9. $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4x+5)^2}$.

17.10. $\int_1^{\infty} \frac{dx}{x\sqrt{x^2+x+1}}$.

II. Investigate the convergence of improper integrals:

17.11. $\int_1^{\infty} \frac{xdx}{\sqrt{x^6+5}}$.

17.12. $\int_1^{+\infty} \frac{\ln x}{x^3}dx$.

17.13. $\int_0^{\infty} \frac{\sqrt{x}\sin(ax)dx}{x^2+b^2};\ a \neq 0;\ b \neq 0$.

17.14. $\int_2^{\infty} \frac{(2+\sin x+\ln x)dx}{\sqrt{x}}$.

17.15. $\int_0^{\infty} \frac{\sin^2 x}{x}dx$.

17.16. $\int_1^{\infty} \frac{dx}{x\sqrt[3]{x^2+1}}$.

17.17. $\int_0^{\infty} \frac{\ln(x+1)}{x^n}dx$.

17.18. $\int_0^{\infty} \frac{\cos(ax)}{1+x^n}dx$.

III. Investigate the absolute and conditional convergence of integrals:

17.19. $\int_0^{+\infty} x^2 \cdot \cos(e^x)dx$.

17.20. $\int_0^{\infty} x^p \sin(x^q)dx, q \neq 0$.

17.21. $\int_0^{\infty} \frac{\sin^3 x}{x}dx$.

17.22. $\int_0^{\infty} \frac{\sin\left(x+\frac{1}{x}\right)}{x^n}dx$.

IV. Find the principal values of improper integrals of the first kind:

17.23. $V.p.\int_{-\infty}^{+\infty} \operatorname{arctg} x dx$.

17.24. $V.p.\int_{-\infty}^{+\infty} x^2 \cdot \operatorname{arctg} x \cdot dx$.

17.25. $V.p.\int_{-\infty}^{+\infty} \sin x dx$.

17.26. $V.p.\int_{-\infty}^{+\infty} x^4 \cdot \operatorname{arctg} x dx$.

17.27. $V.p.\int_0^{\infty} \frac{dx}{x^2-3x+2}$.

17.28. $V.p.\int_{-2}^{+\infty} \frac{dx}{1-x^2}$.

17.29. Calculate the integral $\int_0^{+\infty} \frac{\sin^2 x dx}{x^2}$ using the well-known ***Dirichlet integral***: $\int_0^{+\infty} \frac{\sin x dx}{x} = \frac{\pi}{2}$.

17.30. Calculate the integral $\int_0^\infty \frac{e^{-x}dx}{\sqrt{x}}$ using the well-known ***Poisson integral***: $\int_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{2}$.

17.31. Calculate the integral $\int_0^\infty \frac{(1-e^{-x})dx}{x\sqrt{x}}$ using the well-known ***Poisson integral***: $\int_0^\infty e^{-z^2}dz = \frac{\sqrt{\pi}}{2}$.

17.32. Calculate the integral $\int_0^\infty \frac{e^{-x}dx}{x^{\frac{1}{4}}}$ using the ***gamma function*** $\Gamma(\alpha) = \int_0^\infty e^{-z} z^{\alpha-1}dz$.

17.33. Calculate the integral $\int_0^\infty \frac{(1-e^{-x})dx}{x^{\frac{5}{4}}}$ using the ***gamma function*** $\Gamma(\alpha) = \int_0^\infty e^{-z} z^{\alpha-1}dz$.

17.34. Calculate the mass of the body formed by rotation around the axis OX of the curved trapezoid formed by the lines: $\left\{ y(x) = x^{-\frac{1}{4}}, y=0, x=0, x\to\infty \right\}$. Assume that the density of the body material is equal to: $\rho(x) = \ln(1+x^{-2})$.

Answers and instructions

17.1. $\pi/2-1$. **17.2.** $\pi/108$. **17.3.** $\frac{13}{90}$. **17.4.** $\frac{2\pi\sqrt{3}}{3}$. **17.5.** $\pi(\sqrt{2}-1)$.

17.6. $\frac{\pi}{\sqrt{2}}$. **17.7.** $\frac{1}{5}\ln\left(1+\frac{2}{\sqrt{3}}\right)$. **17.8.** 0. **17.9.** $\frac{\pi}{2}$.

17.10. $\ln\left(1+\frac{2}{\sqrt{3}}\right)$. **17.11.** Convergent. **17.12.** $0,25$.

17.13. Convergent. **17.14.** Divergent. **17.15.** Divergent.
17.16. Convergent. **17.17.** Conditionally convergent $1<n<2$.
17.18. Conditionally convergent under the condition $n>0, a\neq 0$.

17.19. Convergent. **17.20.** Convergent absolutely under the condition $-1 < \frac{p+1}{q} < 0$, convergent conditionally under the condition $0 \le \frac{p+1}{q} < 1$.

17.21. Conditionally convergent.

17.22. *C*onvergent conditionally under the condition $0 < n < 2$.

17.23. 0. **17.24.** 0. **17.25.** 0. **17.26.** 0. **17.27.** $\ln\left(\frac{1}{2}\right)$.

17.28. 0. **17.29.** $\frac{\pi}{2}$. **17.30.** $\sqrt{\pi}$. **17.31.** $2\sqrt{\pi}$. **17.32.** $\Gamma\left(\frac{3}{4}\right)$.

17.33. $4\Gamma\left(\frac{3}{4}\right)$. **17.34.** $2\sqrt{2}\pi^2$.

PRACTICAL LESSON 18
Improper integrals of the second kind

18.1. Basic concepts and definitions. Three types of singular points and three cases of improper integrals

Let us consider ***three cases*** of improper integrals of unbounded functions:

1) functions with a singularity at the point b of the integration interval $[a,b]$;

2) functions with a singularity at the point a of the integration interval $[a,b]$;

3) functions with a singularity in the middle of the integration interval $c \in [a;b]$.

The first case. Let the function $f(x)$ be defined and integrable in any interval $[a,b-\mu] \subset [a,b)$, $(0<\mu<b-a]$, but ***unbounded*** in every interval $[b-\mu,b)$ to the left of the point b. In other words, the function $f(x)$ has a ***discontinuity*** of the ***second kind*** at a point b. The point b is called a ***special point*** of the function $f(x)$ (Fig. 18.1). Consider an integral with a variable upper limit, which is some function of the argument $\mu \in (0,b-a]$:

$$F(\mu)=\int_a^{b-\mu} f(x)dx .$$

Definition. *The limit of the integral* $\int_a^{b-\mu} f(x)dx$ *(or the limit of the function* $F(\mu)$*) at* $\mu \to 0$, $\mu \in (0,b-a]$ *(finite or infinite) is called an* ***improper integral of the second kind*** *on the interval* $[a,b]$ *from an unbounded function* $f(x)$ *with a singularity at a point* $x=b$ *and is denoted by the symbol*

$$\int_a^b f(x)dx=\lim_{\mu\to 0}\int_a^{b-\mu} f(x)dx . \qquad (18.1)$$

If the limit (18.1) is ***finite***, then the integral is said to be ***convergent***, and the function $f(x)$ is integrable on the interval $[a,b]$. If the limit (18.1) is ***infinite or does not exist***, then the integral is said to be ***divergent***, and the ***function is non-integrable***.

The second case. Let the function $f(x)$ be ***unbounded*** at the left end of the segment $[a,b]$ (at $x=a$), in other words, it has a ***discontinuity of the second kind*** at the point $x=a$ (Fig. 18.2).

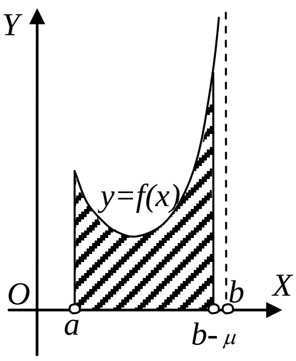

Fig. 18.1

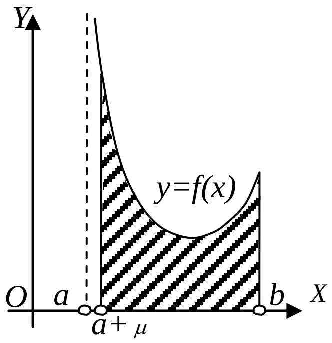

Fig. 18.2

Definition. *An **improper integral of the second kind** on the interval* $[a,b]$ *from an **unbounded function*** $f(x)$ *with a singularity at a point* $x=a$ *is called a finite or infinite **limit** of the integral* $\int\limits_{a+\eta}^{b} f(x)dx$ *at* $\eta \to 0$ $(0<\eta<b-a)$ *and is denoted by the symbol*

$$\int\limits_{a}^{b} f(x)dx = \lim_{\eta \to 0} \int\limits_{a+\eta}^{b} f(x)dx \qquad (18.2)$$

The third case. If the function $f(x)$ has a ***singular point in the middle of the interval*** $x=c\in[a;b]$, then under the condition of the existence of both improper integrals $\int\limits_{a}^{c} f(x)dx$ and $\int\limits_{c}^{b} f(x)dx$ by definition, we put:

$$\int\limits_{a}^{b} f(x)dx = \int\limits_{a}^{c} f(x)dx + \int\limits_{c}^{b} f(x)dx. \qquad (18.3)$$

The improper integral (18.3) should be understood as follows $(0<\mu<c-a, 0<\eta<b-c)$:

$$\int\limits_{a}^{b} f(x)dx = \lim_{\mu \to 0} \int\limits_{a}^{c-\mu} f(x)dx + \lim_{\eta \to 0} \int\limits_{c+\eta}^{b} f(x)dx.$$

In the following, we will consider improper integrals in the form (18.1).

Example 18.1. Calculate the integral $\int\limits_{0}^{1} \frac{dx}{\sqrt{1-x}}$.

The solution. We will show that this integral is convergent:

$$\int\limits_{0}^{1} \frac{dx}{\sqrt{1-x}} = \lim_{\mu \to 0} \int\limits_{0}^{1-\mu} \frac{dx}{\sqrt{1-x}} = -2\lim_{\mu \to 0} \sqrt{1-x}\Big|_{0}^{1-\mu} = -2\left(\lim_{\mu \to 0} \sqrt{\mu} - 1\right) = 2 .$$

Let's introduce the concept of ***absolute and conditional convergence*** for second-order improper integrals.

Definition. *The improper integral (18.1) is called* ***absolutely convergent*** *if the improper* ***integral is convergent***

$$\int_a^b |f(x)|dx \tag{18.4}$$

Definition. *If the integral (18.1) is* ***convergent****, and the integral (18.4) is* ***divergent****, then the integral (18.1) is called* ***conditionally convergent****.*

To establish the necessary and sufficient conditions for the convergence of improper integrals of the second kind, there is a corresponding Cauchy Criterion.

Theorem 18.1 (*Cauchy Criterion*). *In order for the improper integral (18.1) to* ***converge****, it is* ***necessary and sufficient*** *that for an arbitrary number* $\varepsilon > 0$ *it is possible to find a number* $\delta > 0$ *such that for arbitrary numbers* η_1 *and* η_2 *from the interval* $\eta_1, \eta_2 \in (b - \delta, b)$, *the inequality holds:*

$$\left| \int_{\eta_1}^{\eta_2} f(x)dx \right| < \varepsilon . \tag{18.5}$$

The validity of this Criterion stems from the fact that the concept of convergence of the integral (18.1) by definition is equivalent to the concept of the existence of the limit value of the function $F(\mu)$ at $\mu \to 0$, which was introduced at the beginning of this material.

18.2. Comparison signs for improper integrals of the second kind

Theorem 18.2 (*sign of comparison by inequality*). *Let the functions* $f(x)$ *and* $g(x)$ *be defined on the interval* $[a,b)$, *are not bounded on this interval and are* ***integrable*** *on any segment* $[a, b-\mu] \subset [a,b)$, $(0 < \mu < b - a)$. *Then, if the inequality* $0 \le f(x) \le g(x)$ *holds and the improper integral* $\int_a^b g(x)dx$ *is convergent, then* $\int_a^b f(x)dx$ *– is also convergent. And vice versa, from divergence* $\int_a^b f(x)dx$ *comes divergence* $\int_a^b g(x)dx$.

The proof of this Theorem is carried out using the results of Theorem 18.2.

Theorem 18.3 (*limit sign of comparison*). *Let the functions* $f(x)$ *and* $g(x)$ *on the interval* $\left[a;b\right)$ *be* ***continuous, positive and have a singularity*** *at the point* $x=b$. *If there is a* ***finite limit of the ratio*** *of these functions*

$$\lim_{x\to b}\frac{f(x)}{g(x)}=k,\quad 0<k<\infty,$$

then the integrals $\int_a^b f(x)dx$ *and* $\int_a^b g(x)dx$ *are both* ***convergent or divergent at the same time.***

It is possible to transform ***Dirichlet's sign*** regarding the convergence of improper integrals of the second kind from the product of two functions $f(x)$ and $g(x)$. The combination of conditions imposed on the functions $f(x)$ and $g(x)$ is given by the following Theorem.

Theorem 18.4 (*Dirichlet's sign for improper integrals of the second kind*). *If the function* $f(x)$ *is absolutely integrable on the interval* $[a,b]$, *and the function* $g(x)$ *is integrable on* $[a,b]$ *in the usual sense, then the function* $f(x)\cdot g(x)$ *will also be* ***absolutely integrable*** *on this interval.*

Example 18.2. Investigate the convergence of the integral $\int_0^1\frac{dx}{\sqrt{x}+4x^5}$.

The solution. The given integral is convergent, since we have the inequality

$$\forall x\in(0,1]:\quad 0<\frac{1}{\sqrt{x}+4x^5}<\frac{1}{\sqrt{x}},$$

and the ***test integral*** of the function $\frac{1}{\sqrt{x}}$ is ***convergent***: $\int_0^1\frac{dx}{\sqrt{x}}=2$.

Example 18.3. Investigate the convergence of the integral $\int_1^2\frac{\cos x\cdot dx}{\sqrt[3]{x-1}}$.

The solution. Since, under the condition $\forall x\in(1,2]$, there is an inequality $\left|\frac{\cos x}{\sqrt[3]{x-1}}\right|\le\frac{1}{\sqrt[3]{x-1}}$ and the improper integral $\int_1^2\frac{dx}{\sqrt[3]{x-1}}$ is ***convergent***, then the given integral is also ***convergent***, and ***even absolutely convergent***. The same result could be obtained by Dirichlet's sign (Th 4) by noting in the given integral $f(x)=\cos x$ and $g(x)=\frac{1}{\sqrt[3]{x-1}}$.

Example 18.4. Investigate the convergence of the integral $\int_0^1 \frac{dx}{\operatorname{arctg} x}$.

The solution. Consider as a ***test the improper integral*** $\int_0^1 \frac{dx}{x}$, that is ***divergent***. The integral functions $f(x) = \frac{1}{\operatorname{arctg} x}$ and $g(x) = \frac{1}{x}$ these two integrals have a feature at the point $x = 0$. Let us calculate the corresponding limit:

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x}{\operatorname{arctg} x} = 1.$$

Since the test improper integral $\int_0^1 \frac{dx}{x}$ is divergent, the given integral is ***also divergent***.

Theorem 18.5. *If the function $f(x)$ changes sign on the interval $[a, b)$ and is infinite at the point b, and the improper integral $\int_a^b |f(x)| dx$ of the* ***absolute value*** *of this function is* ***convergent****, then the integral $\int_a^b f(x) dx$ is also* ***convergent****.*

Remark. Test functions $\frac{1}{(b-x)^{\alpha}}$ or $\frac{1}{(x-a)^{\alpha}}$ are often chosen as ***comparison functions***. We will show that the corresponding ***test integrals*** $\int_a^b \frac{1}{(b-x)^{\alpha}} dx$ and $\int_a^b \frac{1}{(x-a)^{\alpha}} dx$ both ***converge*** with respect to $\alpha < 1$ and ***diverge*** with respect to $\alpha \geq 1$.

Let us prove the first statement that the test integral $\int_a^b \frac{1}{(b-x)^{\alpha}} dx$ is convergent for $\alpha < 1$:

$$\int_a^{b-\mu} \frac{dx}{(b-x)^{\alpha}} = \begin{cases} \left. \frac{(b-x)^{1-\alpha}}{1-\alpha} \right|_a^{b-\mu} = \frac{(b-a)^{1-\alpha} - \mu^{1-\alpha}}{1-\alpha}, if\ \alpha \neq 1; \\ -\ln(b-x)\Big|_a^{b-\mu} = \ln\left(\frac{b-a}{\mu}\right), if\ \alpha = 1. \end{cases}.$$

It is obvious that the limit $\lim\limits_{\mu\to+0}\int\limits_a^{b-\mu}\frac{1}{(b-x)^{\alpha}}dx$ exists and is equal to $\frac{(b-a)^{1-\alpha}}{1-\alpha}$ under the condition $\alpha<1$ and does not exist under the condition $\alpha\ge 1$. Therefore, the given ***improper test integral*** is ***convergent*** for $\alpha<1$ and ***divergent*** for $\alpha\ge 1$.

Example 18.5. Investigate the convergence of the integral $\int\limits_0^1\frac{1}{\sqrt[3]{x(e^x-e^{-x})}}dx$.

The solution. A singular point of the integrand function is a point $x=0$. We use the previous Remark and set the degree α of the denominator, that is, its order relative to x. To do this, let's represent exponents with Taylor's formulas

$$e^x-e^{-x}=1+x-1+x+\ldots=2x+\ldots,$$

where terms of higher order than the first are indicated by dots. Therefore, the degree of the denominator is equal to $\alpha=\frac{2}{3}<1$. Hence we conclude about the ***convergence*** of the given improper integral.

Example 18.6. Investigate the convergence of the integral $\int\limits_0^1\frac{x^{b-1}-x^{a-1}}{\ln x}dx, a>0, b>0$.

The solution. Under the condition $x\to 1$, we have an uncertainty, after revealing which we will get a finite limit:

$$\lim_{x\to 1}\frac{x^{b-1}-x^{a-1}}{\ln x}=\left\{\frac{0}{0}\right\}=\left|Hospital\right|=\lim_{x\to 1}\frac{(b-1)x^{b-2}-(a-1)x^{a-2}}{\frac{1}{x}}=b-a.$$

Suppose that at least one of the numbers a or b is less than one. Then the point $x=0$ is a singular point of the integrand function. In this case, the ratio of the integral function to the numerator is equal (to the condition $x\to 0$)

$$\frac{1}{\ln x}\to 0.$$

Therefore, since the integral $\int\limits_0^1(x^{b-1}-x^{a-1})dx$ is convergent, the given integral is also ***convergent***.

Remark. If the function $F(x)$ is the primitive of the integral function $f(x)$ and $F(x)$ is ***continuous*** on the segment $[a,b]$, then the Newton–Leibniz formula can be applied to calculate the improper integral of the unbounded function even in the case when ***the singular point*** $f(x)$ ***lies in the middle of the segment*** $[a,b]$. So, to calculate improper integrals in this case, we have a formula of the usual form:

$$\int_a^b f(x)dx = F(b) - F(a). \tag{18.6}$$

Example 18.7. Find the improper integral: $\int_0^{\pi/2} \frac{\cos^3 x dx}{\sqrt{\sin x}}$.

The solution. We find the primitive $F(x)$ of the integral function $f(x)$ and apply formula (18.6):

$$\int_0^{\pi/2} \frac{\cos^3 x dx}{\sqrt{\sin x}} = \lim_{\mu \to 0} \int_\mu^{\pi/2} \frac{\cos^3 x \cdot dx}{\sqrt{\sin x}} = \lim_{\mu \to 0} \int_\mu^{\pi/2} \frac{(1-\sin^2 x)\cdot d(\sin x)}{\sqrt{\sin x}} = \lim_{\mu \to 0} \int_\mu^{\pi/2} \sin^{-\frac{1}{2}} x \cdot d(\sin x) -$$

$$- \lim_{\mu \to 0} \int_\mu^{\pi/2} \sin^{3/2} x \cdot d(\sin x) = 2\sqrt{\sin x}\Big|_0^{\frac{\pi}{2}} - \frac{2}{5}\sin^{5/2} x\Big|_0^{\frac{\pi}{2}} = 2 - \frac{2}{5} = \frac{8}{5}.$$

Example 18.8. Find the improper integral: $\int_0^{33} \frac{dx}{\sqrt[5]{(x-1)^3}}$.

The solution. The function has a ***discontinuity of the second kind*** at the point $x = 1$. We integrate and apply formula (18.6):

$$\int_0^{33} \frac{dx}{\sqrt[5]{(x-1)^3}} = \frac{5}{2}(x-1)^{2/5}\Big|_0^{33} = \frac{5}{2}(4-1) = \frac{15}{2}.$$

18.3. The concept of the principal value of an improper integral of the second kind

Let the function $f(x)$ be ***defined*** on the segment $[a,b]$ except, possibly, the point c, $(a < c < b)$, and ***integrable*** on any segment $[a,\eta] \subset [a,c)$, as well as on any segment $[\mu,b] \subset (c,b]$.

Definition. *If there is a limit*

$$\lim_{\varepsilon \to +0}\left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right),$$

then it is called the ***principal value*** *of the* ***improper integral*** $\int_a^b f(x)dx$ *and is denoted as follows:* $V.p.\int_a^b f(x)dx$.

So, if the function $f(x)$ has a ***singularity*** at the point $c \in [a,b]$, then by the definition of the principal value of the improper integral we have:

$$V.p.\int_a^b f(x)dx = \lim_{\varepsilon \to +0} \left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^b f(x)dx \right).$$

Example 18.9. Find the principal value of the improper integral $\int_a^b \frac{dx}{x-c}$, $a < c < b$.

The solution. It is obvious that the improper integral in the usual sense is ***divergent***. However, the principal value of this integral exists and is ***a finite number***:

$$V.p.\int_a^b \frac{dx}{x-c} = \lim_{\varepsilon \to +0} \left(\int_a^{c-\varepsilon} \frac{dx}{x-c} + \int_{c+\varepsilon}^b \frac{dx}{x-c} \right) = \ln\left(\frac{b-c}{c-a} \right).$$

Tasks for classroom and independent work

I. Calculate improper integrals:

18.1. $\int_{-1}^{0} \frac{e^{1/x}}{x^3}dx$.

18.2. $\int_0^1 \ln x dx$.

18.3. $\int_0^1 \frac{dx}{(2-x)\sqrt{1-x}}$.

18.4. $\int_{-2}^{0} \frac{x^3 dx}{\sqrt{4-x^2}}$.

18.5. $\int_0^{\frac{\pi}{2}} \ln \sin x dx$.

18.6. $\int_0^{\frac{\pi}{2}} \ln \cos x dx$.

18.7. $\int_0^1 x^5 \cdot \ln^3\left(\frac{1}{x}\right)dx$.

18.8. $\int_1^{\sqrt{2}} \frac{dx}{x\sqrt{x^2-1}}$.

18.9. $\int_0^1 \frac{x^7 dx}{\sqrt{1-x^4}}$.

18.10. $\int_0^1 \frac{dx}{\sqrt{e^x-1}}$.

II. Investigate the convergence of improper integrals:

18.11. $\int_{-1}^{0} x^6 \ln(x+1)dx$.

18.12. $\int_0^{\frac{\pi}{4}} \frac{xdx}{\sin x^2}$.

18.13. $\int_0^1 \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$.

18.14. $\int_0^1 \frac{xdx}{e^x + 2\cos\sqrt{x} - 3}$.

18.15. $\int_0^{\pi/2} \frac{\ln(\sin(x))dx}{\sqrt{x}}$.

18.16. $\int_0^1 \frac{\ln x dx}{1-x^2}$.

18.17. $\int_0^1 \frac{\ln(1+\sqrt[3]{x})dx}{e^{\sin x}-1}$.

18.18. $\int_0^1 \frac{(3\sin^2 x + \cos 2x)dx}{\sqrt[5]{(x-1)^3}}$.

18.19. $\int_0^{\pi/2} \sin(\sec x)dx$.

18.20. $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}}$.

III. Investigate the absolute and conditional convergence of integrals:

18.21. $\int_2^3 \frac{\sin x dx}{\sqrt[3]{(x-2)^2}}$.

18.22. $\int_0^{\frac{\pi}{2}} \frac{\cos x dx}{\operatorname{sh}(\sqrt{x})}$.

IV. Find the principal values of improper integrals of the second kind:

18.23. $V.p.\int_{-1}^{e} \frac{dx}{x}$.

18.24. $V.p.\int_0^4 \frac{dx}{x^2-5x+6}$.

18.25. $V.p.\int_{1/e}^{e} \frac{dx}{x\ln x}$.

18.26. $V.p.\int_0^2 \frac{dx}{1-x^2}$.

18.27. Investigate the absolute and conditional convergence and calculate the integral $\int_0^\infty \frac{\cos x dx}{\sqrt{x}}$ using the well-known ***Fresnel integral***:

$$\int_0^\infty \cos(x^2)dx = \int_0^\infty \sin(x^2)dx = \sqrt{\frac{\pi}{8}} .$$

18.28. Investigate the absolute and conditional convergence and calculate the integral $\int_0^\infty \frac{\sin x dx}{\sqrt[3]{x}}$ using the ***gamma function*** $\Gamma(\alpha) = \int_0^\infty e^{-z} z^{\alpha-1} dz$.

18.29. Investigate the absolute and conditional convergence and calculate the integral $\int_0^\infty \frac{\cos x dx}{\sqrt[8]{x}}$ using the ***gamma function*** $\Gamma(\alpha) = \int_0^\infty e^{-z} z^{\alpha-1} dz$.

18.30. Find the limit: $\lim_{x\to\infty} \frac{\int_0^x \sqrt{1+t^4}dt}{x^3}$.

Answers and instructions

18.1. $-2/e$. **18.2.** -1. **18.3.** $\pi/2$. **18.4.** $-16/3$. **18.5.** $-\frac{\pi}{2}\ln 2$.

18.6. $-\frac{\pi}{2}\ln 2$. **18.7.** $1/216$. **18.8.** $\pi/4$. **18.9.** $\frac{1}{3}$. **18.10.** $2\,\mathrm{arctg}(\sqrt{e-1})$.

18.11. Convergent. **18.12.** Divergent. **18.13.** Convergent.
18.14. Divergent. **18.15.** Convergent. **18.16.** Convergent.
18.17. Convergent. **18.18.** Convergent. **18.19.** Convergent.
18.20. Convergent. **18.21.** Absolutely convergent.
18.22. Absolutely convergent. **18.23.** 1.

18.24. $-\ln 3$. **18.25.** 0. **18.26.** $\frac{\ln 3}{2}$.

18.27. Conditionally convergent, $\sqrt{\frac{\pi}{2}}$.

18.28. Conditionally convergent, $\frac{\sqrt{3}}{2}\Gamma\left(\frac{2}{3}\right)$.

18.29. Conditionally convergent, $\sin\left(\frac{\pi}{16}\right)\Gamma\left(\frac{7}{8}\right)$. **18.30.** $\frac{1}{3}$.

Recommended Books

1. Легеза В. П. Математичний аналіз : підручник. У 4-х т. Т. 1 / В. П. Легеза. – Київ : КПІ ім. Ігоря Сікорського, Вид-во «Політехніка», 2019. – 336 с.
2. Легеза В. П. Математичний аналіз : підручник. У 4-х т. Т. 2. / В. П. Легеза. – Київ : КПІ ім. Ігоря Сікорського, Вид-во «Політехніка», 2020. – 396 с.
3. Легеза В. П. Математичний аналіз : зб. задач, реком. як навч. посіб. / В. П. Легеза. – Київ : КПІ ім. Ігоря Сікорського, Вид-во «Політехніка», 2018. – 240 с.
4. В.П.Легеза. Математичний аналіз : практикум, реком. як навч. посіб. У 4-х ч. Ч. 1 / В. П. Легеза, О. М. Нещадим. – Київ : КПІ ім. Ігоря Сікорського, Вид-во «Політехніка», 2022. – 216 с.
5. Легеза В. П. Вища математика : підручник. У 2-х ч. Ч. 1 / В. П. Легеза, М. А. Мартиненко, Ю. І. Іванова. – Київ : Четверта хвиля, 2012. – 368 с.
6. Легеза В. П. Вища математика : підручник. У 2-ч. Ч. 2 / В. П. Легеза, М. А. Мартиненко, Ю. І. Іванова. – Київ : Четверта хвиля, 2014. – 368 с.
7. Дубовик В.П. Вища математика / В. П. Дубовик, І. І. Юрик. – Київ : Вища шк., 1993. – 648 с.
8. Дундученко Л. О. Вища математика. У 2-х т. Т. 1 / Л. О. Дундученко, В. В. Ясінський. – Сан-Франциско – Київ : НТУУ «КПІ»; Політехніка, 2006. – 884 с.
9. Дундученко Л.О. Вища математика. У 2-х т. Т. 2 / Л. О. Дундученко, В. В. Ясінський. – Сан-Франциско – Київ : НТУУ «КПІ»; Політехніка, 2007. – 648 с.
10. Ковальчук Б. Математичний аналіз : навч. посіб. У 3-х ч. Ч. 1 / Б. Ковальчук, Й. Шіпка. – Львів : Вид. центр ЛНУ ім. Івана Франка, 2002. – 270 с.
11. Ковальчук Б. Математичний аналіз : навч. посіб. У 3-х ч. Ч. 2 / Б. Ковальчук, Й. Шіпка. – Львів : Вид. центр ЛНУ ім. Івана Франка, 2002. – 282 с.
12. Овчинников П. П. Вища математика : підручник. У 2-х ч. Ч. 1 : Лінійна і векторна алгебра. Аналітична геометрія. Вступ до математичного аналізу. Диференціальне і інтегральне числення / П. П. Овчинников, Ф. Я. Яремчук, В. М. Михайленко. – Київ : Техніка, 2003. – 600 с.
13. Лавренчук В.П. Вища математика. Загальний курс. Ч. 2. Математичний аналіз і диференціальні рівняння / В. П. Лавренчук,

П. П. Настасієв, О. В. Мартинюк, О. С. Кондур. – Чернівці : Книги XXI ст., 2010. – 556 с.

14. Шкіль М.І. Вища математика : підручник. У 2-х ч. Ч. 1 / М. І. Шкіль, Т. В. Колесник, В. М. Котлова. – Київ : Либідь, 2010. – 592 с.

15. Електронний кампус НТУУ «КПІ». Легеза В. П. Матеріали з дисципліни «Математичний аналіз» [Електрон. ресурс]. – Режим доступу : http://login.kpi.ua

16. Google classroom : Практикум з математичного аналізу (перший семестр) [Електрон. ресурс]. – Режим доступу : https://classroom.google.com/w/MTYzMDc4MjAyMjc5/t/all.

17. Google classroom: Практикум з математичного аналізу (другий семестр) [Електрон. ресурс]. – Режим доступу : https://classroom.google.com/w/MjY4ODc3OTMyOTM5/t/all.

18. Fikhtengol'ts G. M. The Fundamentals of Mathematical Analys (Int. series of Monographs on pure and applied Mathematics) / G. M. Fikhtengol'ts. – Vol. 1. – Elsevier, Pergamon Press, 1965. – 520 p.

19. Fikhtengol'ts G. M. The Fundamentals of Mathematical Analysis (Int. series of Monographs on pure and applied Mathematics) / G. M. Fikhtengol'ts. – Vol. 2. – Elsevier, Pergamon Press, 1965. – 518 p.

20. Ilyin V. A. Fundamentals of mathematical analysis. Part 1 / V. A. Ilyin and E. G. Pozyak. – Mir Publishers, 1982. – 637 p.

21. Ilyin V. A. Fundamentals of mathematical analysis. Part 2 / V. A. Ilyin and E. G. Pozyak. – Mir Publishers, 1982. – 438 p.

22. Piskunov N. Differential and Integral calculus / N. Piskunov. – Vol. 1. – CBS Publishers & Distributors, 2021. – 470 p.

23. Berman G. N. A problem book in mathematical analysis / G. N. Berman. – MTG Learning Media (P) Ltd., New Delhi/Gurgaon, 2017. – 490 p.

24. Demidovich B. P. Problems in Mathematical Analysis / B. P. Demidovich. – Gordon & B., 1969. – 496 p.

25. Courant R. Differential and Integral Calculus / R. Courant. – Vol. 1. – Ishi press international, 2010. – 612 p.

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Легеза Віктор Петрович
Нещадим Олександр Михайлович

Практикум з математичного аналізу

Частина 1

Навчальний посібник

Легеза В. П.
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